

# RANDOM VARIABLES WITH COMPLETELY INDEPENDENT SUBCOLLECTIONS

GEORGE A. KIRKUP

**ABSTRACT.** We investigate the algebra and geometry of the independence conditions on discrete random variables in which we fix some random variables and study the complete independence of some subcollections. We interpret such independence conditions on the random variables as an ideal of algebraic relations. After a change of variables, this ideal is generated by generalized  $2 \times 2$  minors of multi-way tables and linear forms. In particular, let  $\Delta$  be a simplicial complex on some random variables and  $A$  be the table corresponding to the product of those random variables. If  $A$  is  $\Delta$ -independent table then  $A$  can be written as the entrywise sum  $A^I + A^0$  where  $A^I$  is a completely independent table and  $A^0$  is identically 0 in its  $\Delta$ -margins.

We compute the isolated components of the original ideal, showing that there is only one component that could correspond to probability distributions, and relate the algebra and geometry of the main component to that of the Segre embedding. If  $\Delta$  has fewer than three facets, we are able to compute generators for the main component, show that it is Cohen–Macaulay, and give a full primary decomposition of the original ideal.

## 1. INTRODUCTION

**1.1. Set-Theoretic Version of the Main Result.** Let  $X_1, \dots, X_n$  be discrete random variables on the same population. Then there is an  $n$ -dimensional table whose  $(i_1, \dots, i_n)$  entry is the probability of  $X_j = i_j$  for all  $j$ . Given the table of probabilities  $A$  for  $X_1, \dots, X_n$ , and a subset  $\mathcal{J} \subset \{1, \dots, n\}$  it is easy to compute the table,  $A_{\mathcal{J}}$  for  $X_{j_1}, \dots, X_{j_r}$  by summing over the indices not in  $\mathcal{J}$ .

The random variables  $X_1, \dots, X_n$  are called completely independent if the probabilities satisfy

$$\text{Prob}(X_1 = i_1, \dots, X_n = i_n) = \prod_j \text{Prob}(X_j = i_j)$$

for all possible  $i_1, \dots, i_n$ . If  $\Delta$  is any collection of subsets of  $\{1, \dots, n\}$  then we say that an  $n$ -dimensional table is  $\Delta$ -independent if for each  $\mathcal{J} \in \Delta$ ,  $A_{\mathcal{J}}$  is completely independent.

With this notation, the main result of this paper implies

---

*Date:* August 16, 2004.

**Theorem 1.** *If  $A$  is a  $\Delta$ -independent table associated to the product variable  $X_1 \times \cdots \times X_n$  then  $A$  can be written as the (entrywise) sum  $A^I + A^0$  where  $A^I$  is the completely independent table and  $A^0$  is a table whose margins  $A_j^0$  are identically 0 for all  $j \in \Delta$ .*

**1.2. The Algebraic Perspective.** The statement above is not the strongest that can be made. Let  $A = (x_{i_1, \dots, i_n})$  be the generic  $a_1 \times \cdots \times a_n$  table over any field  $\mathbb{K}$  and  $\Delta$  be any collection of subsets of  $\{1, \dots, n\}$ . For each  $j \in \Delta$ ,  $A_j$  is a table whose entries are sums of the variables  $x_{i_1, \dots, i_n}$ . Complete independence of a table,  $B$ , with entries in a ring can be expressed by the ideal,  $I(B)$ , generated by generalized  $2 \times 2$  minors of the table. Therefore,  $\Delta$ -independence of the generic table  $A$  is expressed by the ideal

$$I_\Delta(A) = \sum_{j \in \Delta} I(A_j).$$

Theorem 1 is implied by a knowledge of the minimal primes over  $I_\Delta$ . We prove that there is only one minimal prime,  $P_\Delta$  over  $I_\Delta$  which does not contain the sum of all the variables. Therefore,  $P_\Delta$  is the only minimal prime that corresponds to probability distributions. We parameterize  $P_\Delta$  and give set-theoretic generators for it in terms of the generators of a related toric ideal. In the case in which  $\Delta$  has fewer than three facets, we compute the generators for  $P_\Delta$  and show that it is a perfect ideal.

The other minimal primes over  $I_\Delta$  are also accessible, and we give a fairly complete description of them. Moreover, when  $\Delta$  has fewer than three facets we show that  $I_\Delta$  is a radical ideal.  $I_\Delta$  is not always radical and we also give an example in which  $\Delta$  has four facets and  $I_\Delta$  is not radical.

**1.3. Overview.** In Section 2, we define the principal objects of study and develop the elementary statistical terminology needed for the sequel. Section 3 defines the change of variables which is the foundation for the rest of the exposition.

In Section 4 we show that a related toric ideal is contained in  $P_\Delta$ , and in Section 5 we parameterize  $P_\Delta$  and give set-theoretic generators for it. In Section 6 we treat the other minimal primes over  $I_\Delta$  and show that they can be understood in terms of  $P_{\Delta_i}$  for subcomplexes  $\Delta_i \subset \Delta$ . In Section 7 we use principal radical systems to prove that if  $\Delta$  has three or fewer facets then  $P_\Delta$  is generated by the set-theoretic generators given in Section 4 and is a perfect ideal. We also prove that in the same case,  $I_\Delta$  is radical. Finally, Section 8 ties up the loose ends with an example in which  $I_\Delta$  is not radical, two conjectures and notes on the computational limits encountered.

The main theorems are Theorem 7 and Theorem 23. The change of variables in Section 3 and the toric ideal  $Q_\Delta$  from Section 4 are the key technical points to understand from which Theorem 7 follows. Theorem 23 is an application of principal radical systems.

**1.4. Acknowledgements.** I was fully supported by the Air Force, through a National Defense Science and Engineering Graduate Fellowship. I would like to thank my research advisor, David Eisenbud for all his help and support. I also received substantial help and encouragement from Bernd Sturmfels throughout the project.

## 2. STATISTICS FOR ALGEBRAISTS

**2.1. Random Variables.** A *random variable*  $X$  is a function from a set  $\Omega$ , a population, to a set  $S_X$ , the values of  $X$ . We define

$$\{X = s\} = X^{-1}(s).$$

If  $\Omega$  is finite, we define a new function  $P_X: S_X \rightarrow \mathbb{R}_+$  by

$$P_X(s) = \text{Prob}\{X = s\} = \frac{\text{cardinality}\{X = s\}}{\text{cardinality } \Omega}.$$

$P_X(s)$  can be interpreted as the probability that a randomly selected  $\omega \in \Omega$  will have  $X(\omega) = s$ . A *discrete random variable* is a random variable which takes finitely many values. From now on, all our random variables will be discrete on a finite population. That is,  $\Omega$  and  $S_X$  are both finite.

If  $X_1, \dots, X_n$  are random variables on the same population, then there is a product variable

$$X_1 \times \dots \times X_n: \Omega \rightarrow S_{X_1} \times \dots \times S_{X_n}$$

defined in the obvious way. If  $X_j$  takes  $a_j < \infty$  values, then there is an  $a_1 \times \dots \times a_n$   $n$ -dimensional (real) table

$$A = (x_{i_1, \dots, i_n})$$

whose  $(i_1, \dots, i_n)$  entry is the probability,

$$\text{Prob}\{X_1 \times \dots \times X_n = (i_1, \dots, i_n)\}.$$

**2.2. Marginal Tables and Subcollections of Random Variables.** Suppose we have an  $n$ -dimensional array  $A = (x_{i_1, \dots, i_n})$  of probabilities associated to some random variables  $X_1, \dots, X_n$ . Given any  $\mathcal{J} = \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$  we can define an  $a_{j_1} \times \dots \times a_{j_m}$  array which is the probability array for the random variable  $X_{j_1} \times \dots \times X_{j_m}$ , disregarding the other random variables. Such an array is called an *m-margin* of  $A$ .

To recover the probability of some subcollection of events happening, disregarding the other variables, we need only to sum over the variables we wish to disregard. For example, to disregard the random variable  $X_n$ , consider

$$\text{Prob}\{X_1 \times \dots \times X_{n-1} = (i_1, \dots, i_{n-1})\} = \sum_k x_{i_1, \dots, i_{n-1}, k}.$$

In general, suppose that  $A = (x_{i_1, \dots, i_n})$  is an  $n$ -dimensional array with entries in a ring  $R$ . Let  $\sigma$  be an ordered  $n$ -tuple whose  $j^{\text{th}}$  entry,  $\sigma_j$ , is

either an integer such that  $1 \leq \sigma_j \leq a_j$  or the symbol  $+$ . Let

$$\mathcal{J} = \mathcal{J}(\sigma) = \{j_1, \dots, j_m\} = \{j \mid \sigma_j \neq +\}$$

and define

$$x_\sigma := \sum_{i_j = \sigma_j \text{ if } j \in \mathcal{J}} x_{i_1, \dots, i_n}.$$

For example,  $x_{1,+,3} = \sum_j x_{1,j,3}$ .

This essentially allows us to create the desired array, but we need to index the array correctly. Fix some  $\mathcal{J} = \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$  and numbers  $i_1, \dots, i_m$  such that  $1 \leq i_k \leq a_{j_k}$ . We can define a sequence  $\sigma(\mathcal{J})_{\{i_1, \dots, i_m\}}$  of length  $n$ , by  $\sigma(\mathcal{J})_k = +$  if  $k \notin \mathcal{J}$ , and  $\sigma(\mathcal{J})_{j_k} = i_k$ . Again, let  $A = (x_{i_1, \dots, i_n})$  be an  $n$ -dimensional array with entries in a ring  $R$ . We may define an  $a_{j_1} \times \dots \times a_{j_m}$  array  $A_{\mathcal{J}}$  whose  $(i_1, \dots, i_m)$  entry is  $x_{(\sigma(\mathcal{J})_{i_1, \dots, i_m})}$ . This is an  $m$ -margin of  $A$ , as described above.

Moreover, if  $A$  is an array of probabilities that is associated to random variables  $X_1, \dots, X_n$  and  $\mathcal{J} \subset \{1, \dots, n\}$ , then  $A_{\mathcal{J}}$  is the array of probabilities associated to the random variables  $X_{j_1}, \dots, X_{j_m}$ .

**2.3. Complete Independence and the Segre Variety.** The random variables  $X_1, \dots, X_n$  are called *completely independent* if the identity

$$\text{Prob}\{X_1 \times \dots \times X_n = (i_1, \dots, i_n)\} = \prod_{j=1}^n \text{Prob}\{X_j = i_j\}$$

holds for all values in  $S_{X_1} \times \dots \times S_{X_n}$ . We will study the situation in which certain subcollections of the variables  $X_1, \dots, X_n$  are completely independent.

Likewise an array  $A = (x_{i_1, \dots, i_n})$  with entries in a ring  $R$  will be called *completely independent* if there are elements of  $R$ ,  $\{y_{1,i_1}, y_{2,i_2}, \dots, y_{n,i_n}\}$ , such that the condition

$$(1) \quad x_{i_1, \dots, i_n} = \prod y_{j,i_j}$$

holds for all choices  $(i_1, \dots, i_n)$ .

An algebraic geometer will immediately recognize that (1) implies that the table  $A$  is a point on the Segre variety

$$(2) \quad \mathbb{P}^{a_1-1} \times \dots \times \mathbb{P}^{a_n-1} \subset \mathbb{P}^{\prod a_j-1}.$$

This brings us to the link between statistics and commutative algebra.

**2.4. The Algebraic Definitions.** The Segre embedding is induced by the ring map

$$\begin{aligned} \sigma : \mathbb{Z}[x_{i_1, \dots, i_n}] &\longrightarrow \mathbb{Z}[y_{1,i_1}, y_{2,i_2}, \dots, y_{n,i_n}] \\ x_{i_1, \dots, i_n} &\longmapsto \prod y_{j,i_j} \end{aligned}$$

The kernel of  $\sigma$ , which is the defining ideal of the Segre variety, can be generated by generalized  $2 \times 2$  minors, which we now define.

As usual, let  $A = (x_{i_1, \dots, i_n})$  be an  $n$ -dimensional array with entries in a ring  $R$ . We define a  $2 \times 2$  minor about the  $l^{\text{th}}$  coordinate of  $A$  to be any relation of the form

$$\det \begin{pmatrix} x_{i_1, \dots, i_n} & x_{j_1, \dots, j_{l-1}, i_l, j_{l+1}, \dots, j_n} \\ x_{i_1, \dots, i_{l-1}, j_l, i_{l+1}, \dots, i_n} & x_{j_1, \dots, j_n} \end{pmatrix}.$$

This is an interchange of just the  $l^{\text{th}}$  coordinate. Obviously, the ideal in  $R$  generated by all interchanges of one coordinate will generate the ideal containing all interchanges of an arbitrary number of coordinates. From [H  02, Corollary 1.8], we know that the  $2 \times 2$  minors of an  $n$ -dimensional array generate the defining ideal of the Segre embedding. Thus we define the *Segre relations* to be these generalized  $2 \times 2$  minors.

We can define an  $a_1 \times \dots \times a_n$  table with entries in  $R$  to be a map

$$B: \mathbb{Z}[x_{i_1, \dots, i_n}] \longrightarrow R$$

$$x_{i_1, \dots, i_n} \longmapsto b_{i_1, \dots, i_n}$$

where the  $(i_1, \dots, i_n)$  entry in  $B$  is defined to be  $b_{i_1, \dots, i_n}$ . In this language, the generic table is the identity map.

We have a diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R \otimes_{\mathbb{Z}[x_{i_1, \dots, i_n}]} \mathbb{Z}[y_{1, i_1}, y_{2, i_2}, \dots, y_{n, i_n}] \\ B \uparrow & & \uparrow \\ \mathbb{Z}[x_{i_1, \dots, i_n}] & \xrightarrow{\quad \sigma \quad} & \mathbb{Z}[y_{1, i_1}, y_{2, i_2}, \dots, y_{n, i_n}] \end{array}$$

and we let  $I(B) \subset R$  be the kernel of the top map. This amounts to imposing the Segre relations above on the table  $B$ .

Let  $A$  be the generic  $a_1 \times \dots \times a_n$  table and let  $\Delta$  be a collection of subsets of  $\{1, \dots, n\}$ . Recall the definition of the marginal tables  $A_{\mathcal{J}}$  from Section 2.2. We define the ideal

$$I_{\Delta}(A) = \sum_{\mathcal{J} \in \Delta} I(A_{\mathcal{J}}).$$

That is,  $I_{\Delta}(A)$  is the ideal generated by the generalized  $2 \times 2$  minors of each margin  $A_{\mathcal{J}}$ , when  $\mathcal{J} \in \Delta$ . We give an example at the end of this section.

This is a special case of what are called “independence ideals” in the algebraic statistics literature. See [Stu02,   8.1] for more about independence models and their corresponding ideals. One research paper which gives a discussion of conditional independence of four random variables is [Mat99].  $I_{\Delta}(A)$  should be thought of as the defining ideal of the variety of tables which are completely independent in the margins given by  $\Delta$ . We call a table  $\Delta$ -independent if it lies on the variety defined by  $I_{\Delta}(A)$ .

If  $\mathcal{J}' \subset \mathcal{J}$ , then because of the multilinearity of the Segre relations, the complete independence of  $A_{\mathcal{J}}$  implies the complete independence of  $A_{\mathcal{J}'}$ . Thus we may assume that  $\Delta$  has the structure of a simplicial complex; that is,  $\mathcal{J}' \subset \mathcal{J} \in \Delta \implies \mathcal{J}' \in \Delta$ .

The rest of the paper is concerned with the primary decomposition of the ideals  $I_\Delta(A)$ . For any  $\Delta$  we will show there is only one minimal prime which does not contain  $x_{+, \dots, +}$ . This component is the most important because when  $A$  represents a probability distribution,  $x_{+, \dots, +} = 1$ . Thus we study that prime and relate it algebraically and geometrically to the Segre variety. When  $\Delta$  is a simplicial complex with three or fewer facets, we can compute generators for the main component and show that it is perfect. In that case we will also show that  $I_\Delta(A)$  is a radical ideal and give a full primary decomposition.

Throughout the exposition, we will consider the following running example for clarity:  $n = 3$ ,  $a_1 = a_2 = a_3 = 2$ , and

$$\Delta = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}.$$

In this case  $R = \mathbb{K}[x_{i,j,k}]$  is a polynomial ring with 8 variables and  $I_\Delta$  is generated by 3 elements:

$$I_\Delta = \langle \det \begin{pmatrix} x_{1,1,+} & x_{1,2,+} \\ x_{2,1,+} & x_{2,2,+} \end{pmatrix}, \det \begin{pmatrix} x_{1,+,1} & x_{1,+,2} \\ x_{2,+,1} & x_{2,+,2} \end{pmatrix}, \det \begin{pmatrix} x_{+,1,1} & x_{+,1,2} \\ x_{+,2,1} & x_{+,2,2} \end{pmatrix} \rangle.$$

Despite its appearance,  $I_\Delta$  is not a binomial ideal because  $x_{1,1,+} = x_{1,1,1} + x_{1,1,2}$ .

### 3. A LINEAR CHANGE OF VARIABLES

**3.1. Set-theoretic Heuristics.** Let  $\Delta$  be some fixed collection of subsets of  $\{1, \dots, n\}$ . Our goal is to decompose the ideal  $I_\Delta(A) \subset R = \mathbb{K}[x_{i_1, \dots, i_n}]$  which is defined by the complete independence of the collection of margins of the generic table  $A$  given by  $\Delta$ . First, it will be helpful and illuminating to perform a linear change of variables on  $R$  which makes  $I_\Delta$  an ideal generated by quadratic binomials and linear forms. We will show that  $R/I_\Delta(A)$  is a polynomial ring over a ring of smaller dimension.

Set-theoretically, suppose that one table  $A$  is  $\Delta$ -independent, and another table  $B$  has the property that for each  $\mathcal{J} \in \Delta$ ,  $B_{\mathcal{J}} = 0$ . Then the sum (entry by entry)  $A + B$  is also  $\Delta$ -independent. This is a trivial result of the fact that the equations which define  $\Delta$ -independence only involve entries of the marginal tables and  $B$  is identically 0 in its  $\Delta$ -margins. In this section, we will develop this idea algebraically.

**3.2.  $S_\Delta$ ,  $T_\Delta$  and the Change of Variables.** We define a ring  $S_\Delta$ , to be the polynomial ring over  $\mathbb{K}$  with variables that are indexed by the entries in the marginal tables given by the elements of  $\Delta$ . That is, for every  $\mathcal{J} \in \Delta$ ,  $A_{\mathcal{J}} = (x_{i_1, \dots, i_n})$  with  $i_k = +$  for every  $k \notin \mathcal{J}$ . So for every  $\mathcal{J} \in \Delta$ , create a formal symbol  $X_{i_1, \dots, i_n}$  with  $i_k = \bullet$  for every  $k \notin \mathcal{J}$  and  $1 \leq i_j \leq a_j$  for all  $j \in \mathcal{J}$ . Then let  $S_\Delta$  be the polynomial ring over  $\mathbb{K}$  generated by these formal symbols.

Now consider the map of rings  $\tau_\Delta: S_\Delta \longrightarrow R$  defined by

$$X_{i_1, \dots, i_n} \longmapsto x_{i_1, \dots, i_n}$$

in which  $\bullet$  changes to  $+$ . The kernel of  $\tau_\Delta$ ,  $K_\Delta \subset S_\Delta$ , is generated by linear forms. Let  $T_\Delta = S_\Delta/K_\Delta$  be the coordinate ring of  $\Delta$ -marginal tables. Set-theoretically, a  $\Delta$ -marginal table  $B$  represents the class of tables  $B'$  such that for all  $j \in \Delta$ ,  $B_j = B'_j$ .

If  $\Delta$  and  $\Delta'$  have the property that the maximal elements of  $\Delta$  and  $\Delta'$  are the same, it is clear that  $T_\Delta \cong T_{\Delta'}$ . Since there is no ambiguity in  $T_\Delta$ , we will replace all  $\bullet$ 's in the indices of the variables by  $+$ 's as usual.

On the other hand, let

$$Z_\Delta = R/(L_\Delta(A)),$$

the coordinate ring of tables whose margins given by  $\Delta$  are identically zero. Since the ideal we quotient by is generated by linear forms,  $Z_\Delta$  is a polynomial ring over  $\mathbb{K}$ . Moreover, since the image of  $\tau_\Delta$  is generated by the linear forms which generate  $\sum_{\mathcal{L} \in \Delta} L_{\mathcal{L}}$ , we have

$$(3) \quad T_\Delta \otimes Z_\Delta \cong \text{im } \tau_\Delta \otimes Z_\Delta \cong R.$$

Set-theoretically, this says that the space of  $a_1 \times \cdots \times a_n$  tables is a trivial bundle over the space of  $\Delta$ -marginal tables.

**Proposition 2.** *Suppose that  $I = \langle f_n \rangle$  is any ideal in  $R$  such that the  $f_n$  are written entirely in terms of the margins given by  $\Delta$ , as above. Then let  $F_n$  be the polynomial in  $S_\Delta$  (or  $T_\Delta$ ) which has the same form as  $f_n$  except that the lower-case  $x$ 's are replaced by upper-case  $X$ 's and the  $+$ 's are replaced by  $\bullet$ 's. Let  $I^{S_\Delta} := K_\Delta + \langle F_n \rangle$ .*

*Then  $I$  is prime (respectively radical, perfect) if and only if  $I^{S_\Delta}$  is prime (respectively radical, perfect). Moreover, the Betti diagram of  $I$  as an  $R$ -module is the same as that of  $I^{T_\Delta}$  as a  $T_\Delta$ -module.*

*Proof.* Since polynomial rings are flat over the ground field, by (3)

$$R/I \cong S_\Delta/(I^{S_\Delta}) \otimes Z_\Delta,$$

which is a polynomial ring over  $S_\Delta/(I^{S_\Delta})$ . Thus,  $R/I$  is a domain (resp. reduced, Cohen–Macaulay) if and only if  $S_\Delta/I^{S_\Delta}$  is a domain (resp. reduced, Cohen–Macaulay).  $\square$

**3.3. Generators for  $K_\Delta$  and Our Example.** We can also describe the generators of  $K_\Delta$ . The idea is that if we have two margins  $A_j$  and  $A_K$  then they have an “intersection” which is  $A_{j \cap K}$ . In particular, the entries of  $A_{j \cap K}$  will have a representation as sums of elements of  $A_j$  and  $A_K$ , and they must agree. For ease of notation, we will assume that  $j = \{1, \dots, r\}$  and  $K = \{s, \dots, n\}$ , so  $\mathcal{L} = \{r, \dots, s\}$ . Then we have an ideal of relations

$$R_{j,K} = \left\langle \sum_{i_m | m < s} X_{i_1, \dots, i_r, +, \dots, +} - \sum_{i_m | m > r} X_{+, \dots, +, i_s, \dots, i_n} \right\rangle$$

for all choices of  $(i_s, \dots, i_r)$ .

**Proposition 3.**  *$K_\Delta$  is generated by  $\sum R_{j,K}$  for all pairs of  $j, K \in \Delta$ .*

We now turn to our example, in which  $n = 3$ , and

$$\Delta = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}.$$

In this case,

$$S_\Delta = \mathbb{K}[X_{i,j,\bullet}, X_{i,\bullet,k}, X_{\bullet,j,k}, X_{i,\bullet,\bullet}, X_{\bullet,j,\bullet}, X_{\bullet,\bullet,k}, X_{\bullet,\bullet,\bullet}]$$

for  $1 \leq i, j, k \leq 2$ . Thus  $S_\Delta$  has 19 variables.  $K_\Delta$  is easy to describe. For example,

$$R_{\{1,2\},\{1,3\}} = \langle (X_{1,1,\bullet} + X_{1,2,\bullet}) - (X_{1,\bullet,1} + X_{1,\bullet,2}), (X_{2,1,\bullet} + X_{2,2,\bullet}) - (X_{2,\bullet,1} + X_{2,\bullet,2}) \rangle,$$

$$R_{\{1,2\},\{1\}} = \langle (X_{1,1,\bullet} + X_{1,2,\bullet}) - X_{1,\bullet,\bullet}, (X_{2,1,\bullet} + X_{2,2,\bullet}) - X_{2,\bullet,\bullet} \rangle,$$

$$R_{\{1,2\},\{3\}} = \langle (X_{1,1,\bullet} + X_{1,2,\bullet} + X_{2,1,\bullet} + X_{2,2,\bullet}) - (X_{\bullet,\bullet,1} + X_{\bullet,\bullet,2}) \rangle.$$

$K_\Delta$  can be generated by 30 linear forms, but of course this is not minimal, as illustrated by the inclusion

$$R_{\{1,3\},\{1\}} \subset R_{\{1,2\},\{1,3\}} + R_{\{1,2\},\{1\}}.$$

$K_\Delta$  can be minimally generated by 12 linear forms so  $T_\Delta = S_\Delta/K_\Delta$  is a polynomial ring of dimension 7.

Notice that if  $\Delta' = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ ,  $S_{\Delta'}$  has 12 variables, and  $K_{\Delta'}$  is minimally generated by 5 linear forms, and  $T_{\Delta'} \cong T_\Delta$ .

#### 4. IN SEARCH OF A STATISTICALLY SIGNIFICANT COMPONENT $I_\Delta(A)$

**4.1. A Related Toric Ideal.** In the following sections we will prove that there is only one minimal prime over  $I_\Delta(A)$ , for a generic  $a_1 \times \cdots \times a_n$  table  $A$ , which does not contain  $x_{+, \dots, +}$ . This will be the only statistically significant component of  $I_\Delta$  because when  $A$  is a probability distribution,  $x_{+, \dots, +} = 1$ . We will identify the main component as the kernel of ring map, and relate it to a toric ideal.

The first step is to define the toric ideal. Let  $\Delta$  be a collection of subsets of  $\{1, \dots, n\}$  and let

$$\eta_\Delta: S_\Delta \longrightarrow \mathbb{K}[y_{i,j_i} \mid 1 \leq j_i \leq a_i \text{ or } j_i = \bullet]$$

$$X_{j_1, \dots, j_n} \longmapsto \prod y_{i,j_i}$$

Finally, let  $Q_\Delta = \ker \eta_\Delta$ . Since  $Q_\Delta$  is defined as the kernel of a monomial map, it is generated by binomials. The rest of this section will be devoted to showing that  $Q_\Delta$  is contained in  $(I_\Delta : x_{+, \dots, +}^\infty)$ .

In our example, where  $\Delta$  has facets  $\{1, 2\}, \{1, 3\}, \{2, 3\}$ ,  $Q_\Delta$  is generated by such binomials as

$$\det \begin{pmatrix} X_{1,1,\bullet} & X_{2,1,\bullet} \\ X_{1,\bullet,1} & X_{2,\bullet,1} \end{pmatrix} \text{ and } \det \begin{pmatrix} X_{1,1,\bullet} & X_{2,1,\bullet} \\ X_{1,\bullet,\bullet} & X_{2,\bullet,\bullet} \end{pmatrix}.$$

$S_\Delta/Q_\Delta$  is a ring of dimension 6.



**4.2. Some Useful Elements of the Ideal  $I(A)$ .** First we will construct elements in  $I(A)$  which will allow us to view  $+$  like any other index.

**Proposition 4.** *Let  $A$  be the generic  $a_1 \times \dots \times a_n$  table. Then*

$$\det \begin{pmatrix} x_{i_1, \dots, i_n} & x_{j_1, \dots, j_{l-1}, i_l, j_{l+1}, \dots, j_n} \\ x_{i_1, \dots, i_{l-1}, +, i_{l+1}, \dots, i_n} & x_{j_1, \dots, j_{l-1}, +, j_{l+1}, \dots, j_n} \end{pmatrix} \in I(A).$$

*Proof.* Consider the sum

$$\sum_k \det \begin{pmatrix} x_{i_1, \dots, i_{l-1}, i_l, i_{l+1}, \dots, i_n} & x_{j_1, \dots, j_{l-1}, i_l, j_{l+1}, \dots, j_n} \\ x_{i_1, \dots, i_{l-1}, k, i_{l+1}, \dots, i_n} & x_{j_1, \dots, j_{l-1}, k, j_{l+1}, \dots, j_n} \end{pmatrix}$$

which by the multilinearity of the minors is

$$\det \begin{pmatrix} x_{i_1, \dots, i_{l-1}, i_l, i_{l+1}, \dots, i_n} & x_{j_1, \dots, j_{l-1}, i_l, j_{l+1}, \dots, j_n} \\ \sum_k x_{i_1, \dots, i_{l-1}, k, i_{l+1}, \dots, i_n} & \sum_k x_{j_1, \dots, j_{l-1}, k, j_{l+1}, \dots, j_n} \end{pmatrix}.$$

By the definition of  $+$  notation from Section 2.2, this is

$$\det \begin{pmatrix} x_{i_1, \dots, i_n} & x_{j_1, \dots, j_{l-1}, i_l, j_{l+1}, \dots, j_n} \\ x_{i_1, \dots, i_{l-1}, +, i_{l+1}, \dots, i_n} & x_{j_1, \dots, j_{l-1}, +, j_{l+1}, \dots, j_n} \end{pmatrix}$$

which establishes the result.  $\square$

This proposition allows us to let any number of coordinates equal “+”, and interchange them freely.

As an example, consider the case in which  $n = 2$ . For any  $i_1, i_2$

$$x_{i_1, i_2} x_{+, +} - x_{i_1, +} x_{+, i_2} \in I_{\{1, 2\}}(A).$$

If the  $x_{i,j}$  are really probabilities, then  $x_{+, +} = 1$  so this relation becomes  $x_{i_1, i_2} = x_{i_1, +} x_{+, i_2}$ , which is the independence condition for two random variables, as in (1).

**4.3. An Intermediate Ideal,  $J_\Delta \subset Q_\Delta$ .** There are some quadratic binomials in  $Q_\Delta$  which play a special role in the discussion. Let  $J_\Delta \subset Q_\Delta$  be generated by binomials

$$f = X_{\bar{i}_1} \cdot X_{\bar{i}_2} - X_{\bar{j}_1} \cdot X_{\bar{j}_2} \in Q_\Delta$$

such that  $X_{\bar{i}_1}, X_{\bar{j}_1}$  are both entries in  $A_{\mathcal{J}}$  for some  $\mathcal{J} \in \Delta$ . Since  $f \in Q_\Delta$ , this implies that  $X_{\bar{i}_2}, X_{\bar{j}_2}$  are both entries in  $A_{\mathcal{K}}$  for some  $\mathcal{K} \in \Delta$ .

In our example,  $J_\Delta$  will be generated by  $I_\Delta$  and the  $2 \times 2$  minors of the three matrices symmetric to

$$(4) \quad \begin{pmatrix} X_{1,1,\bullet} & X_{1,2,\bullet} & X_{1,\bullet,\bullet} & X_{1,\bullet,1} & X_{1,\bullet,2} \\ X_{2,1,\bullet} & X_{2,2,\bullet} & X_{2,\bullet,\bullet} & X_{2,\bullet,1} & X_{2,\bullet,2} \end{pmatrix}.$$

**Lemma 5.** *Let  $f = X_{\bar{i}_1} \cdot X_{\bar{i}_2} - X_{\bar{j}_1} \cdot X_{\bar{j}_2}$  be a generator of  $J_\Delta$  such that  $X_{\bar{i}_1}$  is an entry in  $A_{\mathcal{J}}$  and  $X_{\bar{i}_2}$  is an entry in  $A_{\mathcal{K}}$ . Then*

$$L_{\mathcal{J} \cap \mathcal{K}} \cdot \langle f \rangle \subset I_\Delta.$$

*Proof.* The proof is very technical (but elementary). In our running example, the result follows from the following line of reasoning. The matrix (4) has the property that the first 3 columns and the last 3 columns have rank 1. Since they share the middle column, either each column of (4) is a scalar multiple of the middle column, or the middle column is identically 0. Thus, either the  $2 \times 2$  minors of (4) vanish or  $X_{1,\bullet,\bullet} = X_{2,\bullet,\bullet} = 0$ .

Now we turn to the detailed proof. Since all the calculations will happen in the margin  $A_{\mathcal{J} \cup \mathcal{K}}$ , we can assume that  $\{1, \dots, n\} = \mathcal{J} \cup \mathcal{K}$  for ease of notation. We re-index so that  $\mathcal{J} = \{1, \dots, s\}$  and  $\mathcal{K} = \{r, \dots, n\}$ , so  $\mathcal{L} = \{r, \dots, s\}$ . After this reorganization,  $f$  is the following determinant

$$q = \det \begin{pmatrix} X_{i_{1,1}, \dots, i_{1,s}, +, \dots, +} & X_{+, \dots, +, j_{2,r}, \dots, j_{2,n}} \\ X_{j_{1,1}, \dots, j_{1,s}, +, \dots, +} & X_{+, \dots, +, i_{2,r}, \dots, i_{2,n}} \end{pmatrix}$$

where  $i_{1,k} = j_{1,k}$  for all  $k < r$  and  $i_{2,k} = j_{2,k}$  for all  $k > s$ . Moreover,  $\{i_{1,k}, i_{2,k}\} = \{j_{1,k}, j_{2,k}\}$  for each  $r \leq k \leq s$ . Thus,  $f$  can be thought of as the exchange of some number of indices between  $X_{\bar{i}_1}$  and  $X_{\bar{i}_2}$ . Clearly, these exchanges can be generated by exchanges of one coordinate. Re-index again, so that  $f$  is an exchange of the  $r^{\text{th}}$  coordinate. Then  $f$  can be written as

$$f = \det \begin{pmatrix} X_{j_1, \dots, j_{r-1}, j_r, j_{r+1}, \dots, j_s, +, \dots, +} & X_{+, \dots, +, k_r, j_{r+1}, \dots, j_n} \\ X_{j_1, \dots, j_{r-1}, k_r, j_{r+1}, \dots, j_s, +, \dots, +} & X_{+, \dots, +, j_r, k_{r+1}, \dots, k_n} \end{pmatrix}.$$

If  $l = x_{+, \dots, +, i_r, \dots, i_s, +, \dots, +}$  is any generator of  $L_{\mathcal{L}}$ , then we need to show that  $lf$  is in  $I(A_{\mathcal{J}}) + I(A_{\mathcal{K}})$ . We will construct this product explicitly.

Consider the sum

$$\begin{aligned} & x_{j_1, \dots, j_{r-1}, i_r, j_{r+1}, \dots, j_s, +, \dots, +} \det \begin{pmatrix} x_{+, \dots, +, k_r, k_{r+1}, \dots, k_n} & x_{+, \dots, +, k_r, i_{r+1}, \dots, i_s, +, \dots, +} \\ x_{+, \dots, +, j_r, k_{r+1}, \dots, k_n} & x_{+, \dots, +, j_r, i_{r+1}, \dots, i_s, +, \dots, +} \end{pmatrix} + \\ & x_{+, \dots, +, k_r, k_{r+1}, \dots, k_n} \det \begin{pmatrix} x_{j_1, \dots, j_{r-1}, j_r, j_{r+1}, \dots, j_s, +, \dots, +} & x_{+, \dots, +, j_r, i_{r+1}, \dots, i_s, +, \dots, +} \\ x_{j_1, \dots, j_{r-1}, i_r, j_{r+1}, \dots, j_s, +, \dots, +} & x_{+, \dots, +, i_r, i_{r+1}, \dots, i_s, +, \dots, +} \end{pmatrix} + \\ & x_{+, \dots, +, j_r, \dots, j_s, k_{s+1}, \dots, k_n} \det \begin{pmatrix} x_{j_1, \dots, j_{r-1}, i_r, j_{r+1}, \dots, j_s, +, \dots, +} & x_{+, \dots, +, i_r, i_{r+1}, \dots, i_s, +, \dots, +} \\ x_{j_1, \dots, j_{r-1}, k_r, j_{r+1}, \dots, j_s, +, \dots, +} & x_{+, \dots, +, k_r, i_{r+1}, \dots, i_s, +, \dots, +} \end{pmatrix} \end{aligned}$$

which is also evidently equal to

$$(X_{+, \dots, +, i_r, \dots, i_s, +, \dots, +})f = lf \in I(A_{\mathcal{J}}) + I(A_{\mathcal{K}}).$$

This completes the calculation.  $\square$

In our example, in which  $\Delta$  has facets  $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ ,  $J_{\Delta} = Q_{\Delta}$ . This is a result of the fact that  $\Delta$  has three facets. The smallest example in which  $J_{\Delta} \neq Q_{\Delta}$  is when  $\Delta$  has facets

$$\{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}.$$

In this case,

$$X_{1,1,+,+}X_{+,+,1,1} - X_{1,+,1,+}X_{+,1,+,1}$$

is in  $Q_{\Delta}$  but not  $J_{\Delta}$ .

**4.4. The Relationship Between  $Q_\Delta$  and  $I_\Delta$ .** We are ready for the main result of the section.

**Proposition 6.** *Let  $\Delta$  be any simplicial complex, and let  $\mathcal{L}$  be the intersection of the facets of  $\Delta$ . If  $f$  is any binomial of degree  $s$  in  $Q_\Delta$  then*

$$f \in (I_\Delta : L_{\mathcal{L}}^s).$$

*In particular,  $Q_\Delta \subset (I_\Delta : X_{+, \dots, +}^\infty)$ .*

*Proof.* The proof of this proposition is, again, rather technical. The main obstacle is notation which easily gets confusing. Thus, we here give an example which will serve to show how to prove the general result.

As our example, we choose the case in which  $\Delta$  has facets

$$\{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\},$$

and let

$$f = X_{1,1,+,+}X_{+,+,1,1} - X_{1,+,1,+}X_{+,1,+,1}.$$

We will show that  $X_{+,+,+,+}^2 f \in I_\Delta$ . Since  $J_\Delta \subset (I_\Delta : X_{+,+,+,+})$ , it suffices to show that  $X_{+,+,+,+}f \in J_\Delta$ .

$$\begin{aligned} X_{+,+,+,+}(X_{1,1,+,+}X_{+,+,1,1} - X_{1,+,1,+}X_{+,1,+,1}) &= \\ &= X_{+,+,+,+}X_{1,1,+,+}X_{+,+,1,1} - X_{+,+,+,+}X_{1,+,1,+}X_{+,1,+,1} \\ &= X_{1,+,+,+}X_{+,1,+,+}X_{+,+,1,1} - X_{1,+,+,+}X_{+,+,1,+}X_{+,1,+,1} \\ &= X_{1,+,+,+}(X_{+,1,+,+}X_{+,+,1,1} - X_{+,+,1,+}X_{+,1,+,1}) \end{aligned}$$

Since  $X_{+,1,+,+}X_{+,+,1,1} - X_{+,+,1,+}X_{+,1,+,1}$  is in  $J_\Delta$ , the example is shown.  $\square$

## 5. $\Delta$ -INDEPENDENCE AND COMPLETE INDEPENDENCE

**5.1. The Segre Embedding,  $\sigma_\Delta$ , and  $P_\Delta$ .** In this section we study the relationship between tables which are  $\Delta$ -independent and tables which are completely independent. It is obvious that any table which is completely independent is also  $\Delta$ -independent. By Proposition 2, we know that inside the variety of  $\Delta$ -independent tables is a trivial bundle over the Segre variety. We will establish a close connection between the ideal  $K_\Delta + Q_\Delta$  and the defining ideal of the Segre variety. In this section, we will assume that  $\Delta$  is a simplicial complex.

The variety of completely independent tables, or the Segre variety, can be parameterized by

$$\begin{aligned} \sigma_{\{1, \dots, n\}}: R = \mathbb{K}[x_{i_1, \dots, i_n}] &\longrightarrow \mathbb{K}[y_{1, i_1}, \dots, y_{n, i_n}] \\ x_{i_1, \dots, i_n} &\longmapsto \prod_j y_{j, i_j} \end{aligned}$$

as in Section 2.4. This map may be composed with  $\tau_\Delta$  from Section 3.2 to give a map

$$\sigma_\Delta: S_\Delta \longrightarrow \mathbb{K}[y_{1, i_1}, \dots, y_{n, i_n}].$$

Let  $P_\Delta$  be the kernel of  $\sigma_\Delta$ . Thus  $P_\Delta$  is a prime ideal which defines the variety of  $\Delta$ -marginal tables which come from a point on the Segre variety.

We Have the following commutative diagram,

$$\begin{array}{ccccc}
 & & R = \mathbb{K}[x_{i_1, \dots, i_n}] & & \\
 & \nearrow \tau_\Delta & & \searrow \sigma_{\{1, \dots, n\}} & \\
 S_\Delta & & & & \mathbb{K}[y_{1, i_1}, \dots, y_{n, i_n}] \\
 & \searrow \eta_\Delta & & \nearrow \Sigma_\bullet & \\
 & & \mathbb{K}[y_{1, i_1}, \dots, y_{n, i_n} \mid 1 \leq i_j \leq a_i \text{ or } i_j = \bullet] & & 
 \end{array}$$

$\sigma_\Delta$  (horizontal arrow from  $S_\Delta$  to  $\mathbb{K}[y_{1, i_1}, \dots, y_{n, i_n}]$ )

**5.2. The Main Theorem.** We are ready for the main theorem, of which Theorem 1 is a corollary. The above commutative diagram summarizes all the main definitions.

**Theorem 7.** *If  $\Delta$  is any simplicial complex,*

$$P_\Delta = \text{rad}(K_\Delta + Q_\Delta)$$

where  $P_\Delta = \ker \sigma_\Delta$ ,  $K_\Delta = \ker \tau_\Delta$  and  $Q_\Delta = \ker \eta_\Delta$ .

*Proof.* First, we need to show that

$$Q_\Delta + K_\Delta \subset P_\Delta.$$

It suffices to show that  $\sigma_\Delta(Q_\Delta + K_\Delta) = 0$ , which is clear by the definitions.

On the other hand, let  $B$  be any point in  $S_\Delta$  on  $V(K_\Delta + Q_\Delta)$ . Since it is a point on  $V(Q_\Delta)$ , it can be represented by  $b_{i_1, \dots, i_n} = \prod_j y_{j, i_j}$ . Suppose that  $j \in \Delta$  such that  $B_j \neq 0$ . Then re-index so that  $j = \{1, \dots, r\}$  and  $b_{1, \dots, 1, \bullet, \dots, \bullet} \neq 0$ . Now take any  $j \in j$  (and re-index so  $j = 1$ ). Since  $B$  is a point on  $V(K_\Delta)$ ,

$$\begin{aligned}
 y_{1, \bullet} \prod_2^r y_{j, 1} \prod_{r+1}^n y_{j, \bullet} &= b_{\bullet, 1, \dots, 1, \bullet, \dots, \bullet} \\
 &= \sum_1^{a_1} b_{i_1, 1, \dots, 1, \bullet, \dots, \bullet} \\
 &= y_{1, +} \prod_2^r y_{j, 1} \prod_{r+1}^n y_{j, \bullet}.
 \end{aligned}$$

Since  $\prod_2^r y_{j, 1} \prod_{r+1}^n y_{j, \bullet} \neq 0$ , that means that  $y_{1, \bullet} = y_{1, +}$ . Therefore, if  $j$  is any index such that there is a face  $j \in \Delta$  with  $B_j \neq 0$  then  $y_{j, \bullet} = y_{j, +}$ .

Therefore, if for each  $j$  there is a  $j \in \Delta$  such that  $B_j \neq 0$ . Then we can let  $z_{j, i_j} = y_{j, i_j}$  and

$$b_{i_1, \dots, i_n} = \prod y_{j, i_j}$$

since  $y_{j, \bullet} = y_{j, +}$  for all  $j$ . Therefore,  $B$  is in  $V(P_\Delta)$ .

On the other hand suppose that  $\mathcal{K}$  is the maximal set such that for any face  $\mathcal{J} \in \Delta$ , if  $\mathcal{J} \cap \mathcal{K} \neq \emptyset$ ,  $B_{\mathcal{J}} = 0$ . Re-index so that  $\mathcal{K} = \{1, \dots, r\}$ . If there is any face  $\mathcal{J}$  such that  $B_{\mathcal{J}} \neq 0$ ,  $\mathcal{J}$  must be disjoint from  $\mathcal{K}$ . Re-index again so that face is  $\{r+1, \dots, s\}$ , and we have

$$\prod_{j=1}^r y_{j,\bullet} \cdot \prod_{j=r+1}^s y_{j,i_j} \cdot \prod_{j=s+1}^n y_{j,\bullet} \neq 0.$$

Therefore,  $y_{j,\bullet} \neq 0$  for any  $j \leq r$ .

If  $\mathcal{L} \in \Delta$  such that  $\mathcal{L} \cap \mathcal{K} = \emptyset$  then let  $\mathcal{L}' = \mathcal{L} \setminus \mathcal{K}$ . Since  $B_{\mathcal{L}} = 0$ ,  $B_{\mathcal{L}'} = 0$  also. Re-index so that  $\mathcal{L}' = \{r+1, \dots, s\}$ . Then

$$\prod_{j=1}^r y_{j,\bullet} \cdot \prod_{j=r+1}^s y_{j,i_j} \cdot \prod_{j=s+1}^n y_{j,\bullet} = 0$$

for all  $y_{j,i_j}$ ,  $r < j \leq s$ . Therefore, either there is some  $r < j \leq s$  such that  $y_{j,i_j} = 0$  or there is some  $j > s$  such that  $y_{j,\bullet} = 0$ . The former case is impossible since that would mean  $B_{\mathcal{J}} = 0$  for any  $\mathcal{J} \ni j$  which contradicts the maximality of  $\mathcal{K}$ . Therefore, for any  $\mathcal{L}$  which intersects  $\mathcal{K}$ , there is some  $j \notin \mathcal{K} \cup \mathcal{L}$  such that  $y_{j,\bullet} = 0$ . Since  $j \notin \mathcal{K}$  we know that  $y_{j,+} = y_{j,\bullet}$ .

Therefore, let

$$\begin{aligned} z_{j,1} &= y_{j,\bullet} & \text{for } j \in \mathcal{K} \\ z_{j,i_j} &= 0 & \text{for } j \in \mathcal{K}, i_j > 1 \\ z_{j,i_j} &= y_{j,i_j} & \text{for } j \notin \mathcal{K}. \end{aligned}$$

Notice that  $z_{j,+} = y_{j,\bullet}$  for all  $j$ . By the previous paragraph, if  $\mathcal{J} \cap \mathcal{K} \neq \emptyset$  then  $B_{\mathcal{J}} = 0$ . Moreover, if  $\mathcal{J} \cap \mathcal{K} = \emptyset$  and  $b_{i_1, \dots, i_n}$  is a coordinate of  $\mathcal{J}$  then

$$\prod_{j \in \mathcal{J}} z_{j,i_j} = \prod_{j \in \mathcal{J}} y_{j,i_j} \prod_{j \notin \mathcal{J}} y_{j,\bullet} = b_{i_1, \dots, i_n}$$

so  $B \in V(P_{\Delta})$

We have thus shown that  $V(K_{\Delta} + Q_{\Delta}) = V(P_{\Delta})$ , which implies that  $\text{rad}(K_{\Delta} + Q_{\Delta}) = P_{\Delta}$  since  $P_{\Delta}$  is prime.  $\square$

**Corollary 8.**  $P_{\Delta}$  is the only minimal prime over  $I_{\Delta}$  which does not contain  $x_{+, \dots, +}$ .

*Proof.* By Theorem 7,  $P_{\Delta}$  is the only minimal prime over  $K_{\Delta} + Q_{\Delta}$ . Therefore, by Proposition 6,  $P_{\Delta}$  is the only minimal prime over  $I_{\Delta}$  which does not contain  $x_{+, \dots, +}$ .  $\square$

Now we state Theorem 7 in a set-theoretic form, which slightly generalizes Theorem 1.

**Corollary 9.** Let  $\mathbb{K}$  be any field and  $B$  be any table with entries in  $\mathbb{K}$  which is  $\Delta$ -independent. If the sum of the entries in  $B$  is not 0, then  $B$

can be written as the (entrywise) sum  $B^I + B^0$  where  $B^I$  is the completely independent table whose  $(i_1, \dots, i_n)$  entry is

$$\prod_j B_{+, \dots, +, i_j, +, \dots, +}$$

and  $B^0$  is a table whose  $\Delta$ -margins are identically 0.

**5.3. An Application to Computational Statistics.** Suppose  $B$  is any  $a_1 \times \dots \times a_n$  table, which is the probability distribution for a random variable  $X_1 \times \dots \times X_n$  and we want to know which sets of the random variables are completely independent.

If  $B$  is any table, let  $B^I$  be the table whose  $i_1, \dots, i_n$  entry is

$$(B^I)_{i_1, \dots, i_n} = \prod_j B_{+, \dots, +, i_j, +, \dots, +}$$

and let  $B^0 = B - B^I$ , the entrywise difference of  $B$  and  $B^I$ .

There is a simplicial complex  $\Delta(B)$  such that  $\mathcal{J} \in \Delta(B)$  if and only if  $(B^0)_{\mathcal{J}} = 0$ . Therefore, by Corollary 9,  $\Delta(B)$  gives exactly the collection of subsets of  $\{X_1, \dots, X_n\}$  which are completely independent.

This gives an algorithm for determining which collections of the random variables are completely independent which is more efficient than the obvious one which computes each margin and determines if it lies on the Segre variety.

## 6. THE OTHER MINIMAL PRIMES OVER $I_{\Delta}(A)$

**6.1. Some Technical Results.** Having established that  $P_{\Delta}$  is the only minimal prime over  $I_{\Delta}(A)$  not containing  $x_{+, \dots, +}$ , it remains to discuss the minimal primes over  $I_{\Delta}(A)$  which do contain  $x_{+, \dots, +}$ . The following simple, technical result, which explains the interplay between the  $L_{\mathcal{L}}$  and  $I(A_{\mathcal{K}})$ , will be the foundation of the discussion.

**Proposition 10.** *Suppose that  $\mathcal{L}, \mathcal{K}, \mathcal{J}_1, \mathcal{J}_2$  are subsets of  $\{1, \dots, n\}$  such that*

$$\mathcal{K} \supset \mathcal{J}_1 \cup \mathcal{J}_2,$$

$$\mathcal{L} \supset \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{K}.$$

*Then*

$$L_{\mathcal{J}_1} \cdot L_{\mathcal{J}_2} \subset L_{\mathcal{L}} + I(A_{\mathcal{K}}).$$

*Proof.* It is clear that if  $\mathcal{L} \subset \mathcal{L}'$  then  $L_{\mathcal{L}} \subset L_{\mathcal{L}'}$  so we may assume that

$$\mathcal{K} \supset \mathcal{L} = \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{K}.$$

Since all the calculations will be done in  $A_{\mathcal{K}}$ , we will assume that  $\mathcal{K} = \{1, \dots, n\}$ . Then we re-index so that  $\mathcal{J}_1 = \{1, \dots, s\}$  and  $\mathcal{J}_2 = \{r, \dots, n\}$  so  $\mathcal{L} = \{r, \dots, s\}$ .

Let  $x_{b_1, \dots, b_s, +, \dots, +}$  and  $x_{+, \dots, +, c_r, \dots, c_n}$  be arbitrary generators of  $L_{\mathcal{J}_1}$  and  $L_{\mathcal{J}_2}$  respectively. Now consider the following element of  $I(A)$ .

$$\det \begin{pmatrix} x_{b_1, \dots, b_{r-1}, c_r, \dots, c_n} & x_{b_1, \dots, b_s, +, \dots, +} \\ x_{+, \dots, +, c_r, \dots, c_n} & x_{+, \dots, +, b_r, \dots, b_s, +, \dots, +} \end{pmatrix}$$

The result is clear since  $x_{+, \dots, +, b_r, \dots, b_s, +, \dots, +} \in L_{\mathcal{L}}$ .  $\square$

**Corollary 11.** *Suppose that  $\mathcal{L} \subsetneq \mathcal{K}$  and  $Q$  is a prime ideal containing  $L_{\mathcal{L}} + I(A_{\mathcal{K}})$ . Then there is some*

$$\mathcal{L} \subset \mathcal{K}' \subset \mathcal{K}$$

*with  $|\mathcal{K}'| + 1 = |\mathcal{K}|$  such that  $L_{\mathcal{K}'} \subset Q$ .*

*Proof.* We induct on  $|\mathcal{K}| - |\mathcal{L}|$ . If  $|\mathcal{K}| - |\mathcal{L}| > 1$ , we re-index so that  $\mathcal{K} = \{1, \dots, s\}$  and  $\mathcal{L} = \{1, \dots, r\}$  with  $s > r + 1$ . Thus we let  $\mathcal{J}_1 = \{1, \dots, r + 1\}$  and  $\mathcal{J}_2 = \{1, \dots, r, r + 2, \dots, s\}$ . Then we can apply Proposition 10, so either  $L_{\mathcal{J}_1}$  or  $L_{\mathcal{J}_2}$  is in  $Q$ . If  $L_{\mathcal{J}_1}$  is in  $Q$  we are done. If  $L_{\mathcal{J}_2}$  is in  $Q$ , we are in a smaller case, and thus done by induction.  $\square$

**Lemma 12.** *Let  $\Delta = \{\mathcal{J}_1, \dots, \mathcal{J}_m\}$  be any collection subsets of  $\{1, \dots, n\}$  and let  $\mathcal{K} = \cap \mathcal{J}_i$ . Suppose that  $Q$  is a prime containing  $I_{\Delta}(A) + L_{\mathcal{K}}$ . Then for each  $\mathcal{J}_i$  there is some  $\mathcal{J}_j$  such that  $Q$  contains  $L_{\mathcal{J}_i \cap \mathcal{J}_j}$ .*

*Proof.* Without loss of generality, let  $i = 1$ . By Corollary 11, there is a  $\mathcal{K}' \supset \mathcal{K}$  such that  $|\mathcal{K}'| + 1 = |\mathcal{J}_1|$  and  $Q$  contains  $L_{\mathcal{K}'}$ . Re-index so that  $\mathcal{J}_1 = \{1\} \cup \mathcal{K}'$ . Since  $\mathcal{K}' \supset \mathcal{K}$ , there must be at least one  $\mathcal{J}_j$  such that  $1 \notin \mathcal{J}_j$ . Therefore,  $\mathcal{J}_i \cap \mathcal{J}_j \subset \mathcal{K}'$ , so  $Q$  contains  $L_{\mathcal{J}_i \cap \mathcal{J}_j}$ .  $\square$

We now give a lemma which explains the interplay between  $P_{\Delta}$  and  $x_{+, \dots, +}$ .

**Lemma 13.** *Let  $\Delta = \{\mathcal{J}, \mathcal{K}\}$  and  $i \in \mathcal{J} \cap \mathcal{K}$ . Then*

$$(L_{\mathcal{K} \setminus \{i\}}) \cdot L_{\mathcal{J}} \subset P_{\Delta} + L_{\mathcal{J} \setminus \{i\}}.$$

*Proof.* Re-index so that  $i = 1$ . Let  $x_{j_1, \dots, j_n}$  be any generator of  $L_{\mathcal{J}}$  and  $x_{+, k_2, \dots, k_n}$  be any generator of  $L_{\mathcal{K} \setminus \{1\}}$ . Consider the following element of  $J_{\Delta} \subset P_{\Delta}$

$$\det \begin{pmatrix} x_{j_1, \dots, j_n} & x_{j_1, k_2, \dots, k_n} \\ x_{+, j_2, \dots, j_n} & x_{+, k_1, \dots, k_n} \end{pmatrix}.$$

Since  $x_{+, j_2, \dots, j_n} \in L_{\mathcal{J} \setminus \{1\}}$ , the result is clear.  $\square$

The next proposition uses the previous results in this section to show that any minimal prime over  $I_{\Delta}$  is made up of several  $P_{\Delta_i}$ . The  $\Delta_i$  have the property that each facet of  $\Delta$  is in exactly one  $\Delta_i$ .

**Proposition 14.** *Let  $\Delta$  be any simplicial complex and  $\mathcal{F}_1, \dots, \mathcal{F}_m$  its facets. If  $\mathfrak{a}$  is any minimal prime containing  $I_{\Delta}(A)$ , then there is an equivalence relation on the facets of  $\Delta$ ,*

$$\mathcal{F}_i \sim \mathcal{F}_j \iff L_{\mathcal{F}_i \cap \mathcal{F}_j} \not\subset \mathfrak{a}.$$

This equivalence relation gives a partition of the facets of  $\Delta$ ,  $\Delta_1 \coprod \cdots \coprod \Delta_r$  such that  $P_{\Delta_i} \subset \mathfrak{a}$  for all  $i$ . Moreover, for each  $i$ , there is some set  $\mathcal{J} \subset \cap \Delta_i$  such that  $\mathcal{J} \not\subset \mathcal{F}$  for any facet of  $\Delta$  not in  $\Delta_i$ , and  $\mathfrak{a}$  contains  $L_{\mathcal{F} \setminus \{j\}}$  for each  $\mathcal{F} \in \Delta_i$  and  $j \in \mathcal{J}$ .

*Proof.* It is clear that the relation given is symmetric. Reflexivity relies on the minimality of  $Q$ . If  $Q$  is any prime containing  $I_\Delta + L_{\mathcal{F}_i}$  for any facet  $\mathcal{F}_i$ , then in  $T_\Delta$ ,  $Q$  can be expressed as  $Q' + L_{\mathcal{F}_i}$  where  $Q'$  is an ideal whose generators are written entirely in terms of the facets  $\mathcal{F}_j$ ,  $j \neq i$ . From this perspective, it is clear that

$$Q' + \sum_{j \neq i} L_{\mathcal{F}_i \cap \mathcal{F}_j}$$

is also prime, so  $Q$  was not minimal.

Transitivity of the relation follows easily from Lemma 12. Suppose that  $Q$  contains neither  $L_{\mathcal{F}_i \cap \mathcal{F}_j}$  nor  $L_{\mathcal{F}_i \cap \mathcal{F}_k}$ . Then applying Lemma 12 to the collection  $\{\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k\}$ , we conclude that  $Q$  does not contain  $L_{\mathcal{F}_i \cap \mathcal{F}_j \cap \mathcal{F}_k}$ . Therefore, it cannot contain  $L_{\mathcal{F}_i \cap \mathcal{F}_k}$ .

Let  $\Delta_i$  be any collection of facets such that for  $\mathcal{F}_j, \mathcal{F}_k \in \Delta_i$ ,  $L_{\mathcal{F}_j \cap \mathcal{F}_k} \not\subset Q$ . Let  $\mathcal{K} = \cap_{\mathcal{F} \in \Delta_i} \mathcal{F}$ . By Lemma 12,  $L_{\mathcal{K}} \not\subset Q$ . Therefore, by Proposition 6,  $P_{\Delta_i} \subset Q$ .

Finally, the last statement is a consequence of the definition of the equivalence relation, Corollary 11 and Lemma 13.  $\square$

**6.2. Classification of the Other Minimal Primes.** Next we will show that certain ideals of the kind mentioned in Proposition 14 are actually prime. If  $\Delta$  is a simplicial complex all of whose facets contain the vertex  $k$ , let  $\Delta \setminus k$  be the simplicial complex whose facets are  $\mathcal{J} \setminus \{k\}$  for each facet  $\mathcal{J} \in \Delta$ .

**Theorem 15.** *Let  $\Delta$  be any simplicial complex on  $\{1, \dots, n\}$  and let  $\Delta_1 \coprod \cdots \coprod \Delta_r$  be a partition of the facets of  $\Delta$ .*

*For each  $\Delta_i$  suppose there is a set  $\mathcal{K}_i \subset \cap \Delta_i$  such that for any facet  $\mathcal{J} \in \Delta$  which is not in  $\Delta_i$ ,  $\mathcal{K}_i \setminus \mathcal{J}$  is nontrivial. Then*

$$\mathfrak{a} = \sum_i P_{\Delta_i} + \sum_i \sum_{k \in \mathcal{K}_i} L_{\Delta_i \setminus k}$$

*is a prime ideal.*

*Any minimal prime over  $I_\Delta$  has the form of one of these ideals.*

*Proof.* By Proposition 2, we can show this in  $S_\Delta$ . Notice that if  $\mathcal{J}_i \in \Delta_i$  and  $\mathcal{J}_j \in \Delta_j$ ,  $i \neq j$ , then  $\mathfrak{a} \supset L_{\mathcal{J}_i \cap \mathcal{J}_j}$ . Therefore,  $\mathfrak{a}$  can be expressed as  $\mathfrak{a}_1 + \cdots + \mathfrak{a}_r$  where

$$\mathfrak{a}_i = P_{\Delta_i} + \sum_{k \in \mathcal{K}_i} L_{\Delta_i, \hat{k}}$$

is an ideal in  $S_\Delta$  which is expressed only in terms of the variables in  $S_{\Delta_i}$ . Therefore,

$$S_\Delta / \mathfrak{a} = S_{\Delta_1} / \mathfrak{a}_1 \otimes \cdots \otimes S_{\Delta_r} / \mathfrak{a}_r.$$



Since  $\mathcal{K}_i \subset \Delta_i$ ,  $S_{\Delta_i}/\mathfrak{a}_i$  is an integral domain for each  $i$ . This statement is true regardless of the field of definition. Therefore,  $S_{\Delta_i}/\mathfrak{a}_i$  remains an integral domain when it is tensored with the algebraic closure of  $\mathbb{K}$ . Thus the tensor product  $S_{\Delta}/\mathfrak{a}$  is an integral domain, so  $\mathfrak{a}$  is prime.

The fact that every minimal prime is of this form is a consequence of Proposition 14.  $\square$

**6.3. The Case in Which  $\Delta$  Is a Graph.** Now we will give some special cases of Theorem 15. The first is in the case in which each facet of  $\Delta$  has two elements. In this case,  $\Delta$  is a graph.

We need one preliminary definition. For any  $j$ , let

$$\Delta(j) = \{\mathcal{J} \in \Delta \mid j \in \mathcal{J}\}.$$

**Corollary 16.** *Let  $\Delta$  be any graph. Any minimal prime over  $I_{\Delta}$  is either  $P_{\Delta}$  or can be expressed as*

$$\sum_{j \in \Gamma} P_{\Delta(j)} + \sum_{j \notin \Gamma} L_{\{j\}}$$

for some vertex cover  $\Gamma$  of  $\Delta$ .

*Proof.* This is a direct application of Theorem 15. Since for each facet  $\{i, j\}$  of  $\Delta$ , any prime containing  $I_{\Delta} + \langle x_{+, \dots, +} \rangle$  must contain either  $L_{\{i\}}$  or  $L_{\{j\}}$ , the statement about  $\Gamma$  being a vertex cover follows.  $\square$

**6.4. The Case in Which  $\Delta$  Has Two Facets and Our Example.** The second special case we give is when  $\Delta$  has only two facets.

**Corollary 17.** *If  $\Delta$  is a simplicial complex with two facets,  $\mathcal{J}_1, \mathcal{J}_2$  then the minimal primes over  $I_{\Delta}$  are  $P_{\Delta}$  and*

$$I_{\Delta} + L_{\mathcal{J}_1 \setminus i_1} + L_{\mathcal{J}_2 \setminus i_2}$$

where  $i_1 \notin \mathcal{J}_2$  and  $i_2 \notin \mathcal{J}_1$ .

Finally, we give our running example.

**Corollary 18.** *Let  $\Delta$  have facets  $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ . Then the minimal primes over  $I_{\Delta}$  are*

$$\begin{aligned} P_{\Delta} \\ Q_0 &= I_{\Delta} + L_{\{1\}} + L_{\{2\}} + L_{\{3\}} \\ Q_1 &= I_{\{2,3\}} + P_{\{1,2\}, \{1,3\}} + L_{\{2\}} + L_{\{3\}} \\ Q_2 &= I_{\{1,3\}} + P_{\{1,2\}, \{2,3\}} + L_{\{1\}} + L_{\{3\}} \\ Q_3 &= I_{\{1,2\}} + P_{\{1,3\}, \{2,3\}} + L_{\{1\}} + L_{\{2\}} \end{aligned}$$

unless one of the  $a_i = 2$ , in which case  $Q_0$  is not minimal.

## 7. PRINCIPAL RADICAL SYSTEMS AND TABLES

**7.1. Principal Radical Systems In General.** In this section we will show that if  $\Delta$  is a simplicial complex with three or fewer facets,  $P_\Delta = K_\Delta + Q_\Delta$ , which is a prime, perfect ideal and  $I_\Delta(A)$  is radical. For each of these results we will use principal radical systems.

The notion of a principal radical system has proved very useful in the study of determinantal ideals. Hochster and Eagon developed it as a method for showing that any ideal of minors of a generic matrix was radical. We follow the presentation Bruns and Vetter [BV88, §12].

The main idea is to prove that an ideal is radical by adding in, one at a time, well-selected elements of the ring until we have an ideal which is obviously radical. We will now cite the theorem as stated in [BV88, §12].

**Theorem 19.** *Let  $R$  be a noetherian ring, and  $\mathcal{F}$  a family of ideals in  $R$ . Suppose that for every member  $I \in \mathcal{F}$  which is not known to be radical, there is some  $x \in R$  such that  $I + \langle x \rangle \in \mathcal{F}$  and one of the following conditions holds:*

- (1)  $x$  is not a zero-divisor modulo  $\text{rad } I$  and  $\cap_1^\infty (I + \langle x^i \rangle) / I = 0$ .
- (2) there exists an ideal  $J \in \mathcal{F}$ ,  $J \supsetneq I$ , such that  $xJ \subset I$  and  $x$  is not a zero-divisor modulo  $\text{rad } J$ .

*Then all the ideals  $I \in \mathcal{F}$  are radical.*

Note that since all of our rings are graded,  $\cap_1^\infty (I + \langle x^i \rangle) / I = 0$  will automatically be satisfied by the Krull Intersection Theorem. We now apply principal radical systems to the ideals  $P_\Delta$ , starting with the simplest case, when  $\Delta$  has 1 facet.

## 7.2. The Radicality of $K_\Delta + Q_\Delta$ .

**Lemma 20.** *Let  $A$  be the generic  $a_1 \times \cdots \times a_n$  table and let  $\Gamma$  be any collection of subsets of  $\{1, \dots, n\}$ . Then  $I(A) + L_\Gamma$  is radical.*

*Proof.* We induct on  $(a_1, \dots, a_n)$ . The base case is that in which  $a_i = 1$  for all  $i$ . In this case, the polynomial ring is  $\mathbb{K}[x_{1,\dots,1}]$ , which is to say it is has only one variable. If  $L$  is non-empty, then the ideal  $L_\Gamma$  is generated by  $x_{1,\dots,1}$  and if  $L$  is empty, the ideal  $I(A) + L_\Gamma$  is 0.

For any other  $(a_1, \dots, a_n)$ , consider the following families of ideals.

$$F_{l_1, \dots, l_n} = I(A) + L_\Gamma + \langle x_{i_1, \dots, i_n} \mid (i_1, \dots, i_n) \leq_{\text{revlex}} (l_1, \dots, l_n) \rangle$$

$$G_{l_1, \dots, l_n} = I(A) + L_\Gamma + \langle x_{i_1, \dots, i_n} \mid i_j < l_j \text{ for some } j \rangle$$

$G_{l_1, \dots, l_n}$  is radical by induction if any  $l_i > 1$ . Of course,  $G_{1, \dots, 1} = I(A) + L_\Gamma$ . On the other hand, consider any  $l = (l_1, \dots, l_{r-1})$ . Let  $s(l)$  be the least  $l'$  such that  $l' > l$ . Let  $j$  be the least  $j$  such that  $l_j \neq a_j$ . Then

$$s(l) = (1, \dots, 1, l_j + 1, l_{j+1}, \dots, l_{r-1}).$$

By definition,  $F_l + \langle x_{s(l)} \rangle = F_{s(l)}$ . Moreover,  $G_{s(l)} \supsetneq F_l$  unless  $l = (a_1, \dots, a_{n-1}, i)$ , in which case  $F_l = G_{s(l)}$  and is thus radical.

To show that  $x_{s(l)}G_{s(l)} \subset F_l$ , let  $x_{i_1, \dots, i_n}$  be an arbitrary generator of  $G_{s(l)}$  which is not contained in  $I(A) + L_\Gamma$ . By the definition of  $G_{s(l)}$  there is some  $j$  such that  $i_j < s(l)_j$ . By re-indexing, assume  $j = 1$  for ease of notation. The following minor is in  $I(A)$

$$\det \begin{pmatrix} x_{s(l)_1, \dots, s(l)_n} & x_{s(l)_1, i_2, \dots, i_n} \\ x_{i_1, s(l)_2, \dots, s(l)_n} & x_{i_1, i_2, \dots, i_n} \end{pmatrix}.$$

Since  $(i_1, s(l)_2, \dots, s(l)_n) <_{\text{revlex}} s(l)$ ,  $(i_1, s(l)_2, \dots, s(l)_n) \leq_{\text{revlex}} l$ . Therefore, the antidiagonal product is in  $F_l$ , and since the minor is in  $I(A) \subset F_l$ , the diagonal product is also in  $F_l$ .

All that remains to show, then, is that  $x_{s(l)}$  is a nonzerodivisor modulo  $\text{rad } G_{s(l)}$ . Since  $R/G_{s(l)}$  is isomorphic to  $R/(I(A) + L_\Gamma)$  for smaller values of the  $a_i$ , this part is reduced to showing that  $x_{1, \dots, 1}$  is a nonzerodivisor modulo  $\text{rad}(I(A) + L_\Gamma)$ . The minimal primes over  $I(A) + L_\Gamma$  are  $I(A) + L_{\Gamma'}$  where  $\Gamma'$  is a collection of subsets of  $\{1, \dots, n\}$ , each of size  $(n-1)$  and such that every set in  $\Gamma$  is contained in a set in  $\Gamma'$ . These are prime because  $R/(I(A) + L_{\Gamma'})$  is isomorphic to  $R/I(A)$ , again for smaller values of the  $a_i$ . Since  $x_{1, \dots, 1}$  is not in any of the minimal primes, it is a nonzerodivisor modulo  $\text{rad}(I(A) + L_\Gamma)$ .

Therefore, we have shown that  $\{F_{l_1, \dots, l_n}, G_{l_1, \dots, l_n}\}$  is a principal radical system, so  $I(A) + L_\Gamma$  is radical.  $\square$

This relatively simple case actually is very similar to the more complicated cases. We will see very similar arguments again.

**Proposition 21.** *Let  $\Delta$  be a simplicial complex with two facets,  $\mathcal{J}_1, \mathcal{J}_2$  and let  $\mathcal{K}$  be a subset of  $\mathcal{J}_1 \cup \mathcal{J}_2$ . Then the ideal  $K_\Delta + Q_\Delta + L_\mathcal{K}$  is radical.*

*Proof.* We re-index so that  $\mathcal{J}_1 = \{1, \dots, s\}$  and  $\mathcal{J}_2 = \{r, \dots, n\}$ .

If  $\mathcal{K}$  contains  $\mathcal{J}_1$  or  $\mathcal{J}_2$ , this reduces to Lemma 20, so we suppose that  $\mathcal{K}$  contains neither  $\mathcal{J}_1$  nor  $\mathcal{J}_2$ . The minimal primes over  $K_\Delta + Q_\Delta + L_\mathcal{K}$  are all of the form  $K_\Delta + Q_\Delta + L_{\Delta_{\hat{i}}}$  for some  $r \leq i \leq s$  or  $K_\Delta + Q_\Delta + L_{\mathcal{J}_1 \setminus i} + L_{\mathcal{J}_2 \setminus j}$  for some  $i < r$  and  $j > s$ . This implies that  $x_{1, \dots, 1, +, \dots, +}$  is a nonzero divisor modulo  $\text{rad}(K_\Delta + Q_\Delta + L_\mathcal{K})$ .

Consider the following families of ideals.

$$F_{l_1, \dots, l_s} = K_\Delta + Q_\Delta + L_\mathcal{K} + \langle x_{i_1, \dots, i_s, +, \dots, +} \mid (i_1, \dots, i_s) \leq_{\text{revlex}} (l_1, \dots, l_s) \rangle$$

$$G_{l_1, \dots, l_s} = K_\Delta + Q_\Delta + L_\mathcal{K} + \langle x_{i_1, \dots, i_n} \mid i_j < l_j \text{ for some } j \leq s \rangle$$

The  $G_{l_1, \dots, l_s}$  are defined to allow any of the  $i_j = +$ , so long as one of the  $i_j$  is a number and  $i_j < l_j$ .

As in the proof of Lemma 20, we can induct on  $(a_1, \dots, a_n)$ , and thus we can assume that  $G_{l_1, \dots, l_s}$  as long as one of the  $l_i > 1$ . In fact, the entire argument from Lemma 20 is valid. We only need to note that for any  $l$ ,  $x_{s(l)}G_{s(l)} \subset F_l$  and  $F_{a_1, \dots, a_s}$  is radical by Lemma 20.  $\square$

**Theorem 22.** *Suppose that  $\Delta$  is a simplicial complex with no more than 3 facets. Then  $K_\Delta + Q_\Delta$  is radical, hence  $K_\Delta + Q_\Delta = P_\Delta$ .*

*Proof.* If  $\Delta$  has two or fewer facets, then Proposition 21 and Lemma 20 apply. Suppose that  $\Delta$  has facets  $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ , and re-index so that  $\mathcal{J}_1 = \{1, \dots, s\}$ .

As in the previous two proofs, consider the following families of ideals.

$$F_{l_1, \dots, l_s} = K_\Delta + Q_\Delta + \langle x_{i_1, \dots, i_s, +, \dots, +} \mid (i_1, \dots, i_s) \leq_{\text{revlex}} (l_1, \dots, l_s) \rangle$$

$$G_{l_1, \dots, l_s} = K_\Delta + Q_\Delta + \langle x_{i_1, \dots, i_n} \mid i_j < l_j \text{ for some } j \leq s \rangle$$

They form a principal radical system for the following reasons.  $G_{l_1, \dots, l_s}$  is radical by induction on  $(a_1, \dots, a_n)$ .  $F_{1, \dots, 1}$  satisfies condition 1 of Theorem 19 because the radical of  $K_\Delta + Q_\Delta$  is prime. For  $(1, \dots, 1) \leq l \leq (a_1, \dots, a_s)$ ,  $F_l$  satisfies condition 2 of Theorem 19 because  $F_{s(l)} = F_l + \langle x_{s(l), +, \dots, +} \rangle$  and  $x_{s(l), +, \dots, +} \cdot G_{s(l)} \subset F_l$  while  $G_{s(l)} \not\subset F_l$ . Finally,  $F_{a_1, \dots, a_s}$  is radical by Proposition 21.

Therefore, we have shown that  $K_\Delta + Q_\Delta$  is prime if  $\Delta$  has three or fewer faces.  $\square$

**7.3. The Perfection of  $P_\Delta$ .** We now use the preceding proofs to establish more about the algebraic structure of  $P_\Delta$ . In particular, if  $\Delta$  has three or fewer facets, we can show that it is perfect.

**Theorem 23.** *If  $\Delta$  is a simplicial complex with three or fewer facets, then  $P_\Delta$  is a perfect ideal of grade  $1 - n + \sum a_i$ .*

*Proof.* We use Proposition 2 to reduce to showing that  $P_\Delta$  is perfect in the ring  $T_\Delta$ . Throughout this proof we will use the same notation as in the previous proof, and treat all ideals as ideals in  $T_\Delta$ .

The main tool we will use is that if  $M_1, M_2$ , and  $M_3$  are  $R$ -modules such that

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact and  $M_2$  and  $M_3$  are Cohen–Macaulay of depth  $d$  and  $d - 1$  respectively then  $M_1$  is a Cohen–Macaulay module of depth  $d$ .

As usual we prove the result by induction on  $(a_1, \dots, a_n)$  since if all but two of these are 1, the ideal is just the  $2 \times 2$  minors of a generic matrix, for which this theorem is well-known.

We re-index as in the beginning of the proof of Theorem 22, and use its notation. By that proof we know that  $F_{1, \dots, 1}$  is radical. Any prime over  $F_{1, \dots, 1}$  contains either  $P_\Delta + L(A_{\mathcal{J}_1})$  or of a prime of the form

$$H_k = P_\Delta + \langle x_{i_1, \dots, i_n} \mid i_k = 1 \rangle$$

for some  $k < s$ . Let

$$G_l = \cap_{k=1}^l H_k.$$

We will show that  $R/G_s$  has depth  $(\sum a_i) - n$  by induction.  $R/G_1$  is isomorphic to  $R/P_\Delta$  with  $a_1$  reduced by 1. Therefore,  $R/G_1 = R/H_1$  is Cohen–Macaulay of depth

$$1 - n + a_1 - 1 + \sum_2^n a_i = (\sum a_i) - n.$$

Now suppose that we have shown that  $R/G_k$  has depth  $(\sum a_i) - n$  for any choice of  $a_1, \dots, a_n$ . Then there is an exact sequence

$$0 \longrightarrow R/G_{k+1} \longrightarrow R/G_k \oplus R/H_{k+1} \longrightarrow R/(G_k + H_{k+1}) \longrightarrow 0.$$

the last term is isomorphic to  $R/G_k$  where  $a_{k+1}$  is replaced by  $a_{k+1} - 1$ . Thus, it has depth  $(\sum a_i) - n - 1$  by induction. Both summands of the middle term have depth  $(\sum a_i) - n$  by induction. Therefore,  $R/G_{k+1}$  has  $(\sum a_i) - n$ . This implies that  $R/G_s$  is Cohen–Macaulay with depth  $(\sum a_i) - n$  as claimed.

If there is some index  $j \in \mathcal{J}_1$  but  $j \notin \mathcal{J}_l$  for any  $l > 1$  then the only minimal primes over  $F_{1,\dots,1}$  are the  $H_k$ . Since  $F_{1,\dots,1}$  is radical, we know that  $F_{1,\dots,1} = G_r$ . Thus the previous paragraph implies that  $R/F_{1,\dots,1}$  is Cohen–Macaulay of depth  $(\sum a_i) - n$ , and since

$$F_{1,\dots,1} = P_\Delta + \langle x_{1,\dots,1,+,\dots,+} \rangle$$

and since  $P_\Delta$  is prime,  $x_{1,\dots,1,+,\dots,+}$  is a non-zerodivisor modulo it. Thus  $R/P_\Delta$  is Cohen–Macaulay of depth  $1 - n + \sum a_i$ . Note that if  $\Delta$  has two facets (or one), then since neither facet can contain the other, this paragraph implies the theorem for  $P_\Delta$ .

On the other hand, suppose that there is no  $j \in \mathcal{J}_1$  such that  $j \notin \mathcal{J}_l$  for any  $l \neq 1$ . This implies that  $\Delta$  has three facets,  $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ . We may assume that the condition holds for  $\mathcal{J}_2, \mathcal{J}_3$  as well, so for each  $j \in \{1, \dots, n\}$ ,  $j$  is an element of two of  $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ . Therefore,  $\mathcal{J}_1$  must contain the symmetric difference of  $\mathcal{J}_2$  and  $\mathcal{J}_3$ ,  $(\mathcal{J}_2 \cup \mathcal{J}_3) \setminus (\mathcal{J}_2 \cap \mathcal{J}_3)$ . Thus the minimal primes over  $P_\Delta + L_{\mathcal{J}_1}$  are

$$D_i = P_\Delta + L_{\mathcal{J}_1} + L_{\Delta, \hat{i}}$$

for each  $i$  in  $(\mathcal{J}_2 \cap \mathcal{J}_3) \setminus \mathcal{J}_1$ . The  $D_i$  are prime because if  $i \in (\mathcal{J}_2 \cap \mathcal{J}_3) \setminus \mathcal{J}_1$ ,  $R/D_i$  is isomorphic to  $R/(P_{\mathcal{J}_2, \mathcal{J}_3})$  with  $a_i$  reduced by 1. Thus, these prime ideals are also perfect of grade  $(\sum a_i) - n$  by induction. Our next goal is to prove that their intersection is also perfect.

Re-index so that

$$(\mathcal{J}_2 \cap \mathcal{J}_3) \setminus \mathcal{J}_1 = \{1, \dots, m\}$$

and let

$$E_l = \cap_{i=1}^l D_i.$$

Suppose that  $E_k$  is perfect of grade  $(\sum a_i) - n$ . Then we have an exact sequence

$$0 \longrightarrow R/E_{k+1} \longrightarrow R/E_k \oplus R/D_{k+1} \longrightarrow R/(E_k + D_{k+1}) \longrightarrow 0.$$

We know that  $R/E_k$  and  $R/D_{k+1}$  are both Cohen–Macaulay of depth  $(\sum a_i) - n$ , and since  $R/(E_k + D_{k+1}) \cong (R/D_{k+1})/E_k$  which is isomorphic to  $R/E_k$  with  $a_{k+1}$  decreased by 1,  $R/(E_k + D_{k+1})$  is Cohen–Macaulay of depth  $(\sum a_i) - n - 1$ . Therefore, we know that  $R/E_{k+1}$  is Cohen–Macaulay of depth  $(\sum a_i) - n$ . Therefore, by induction,  $(P_\Delta + L_{\mathcal{J}_1})$  is perfect of grade  $(\sum a_i) - n$ .

Finally, we need to show that

$$F_{1,\dots,1} = (P_\Delta + L_{\mathcal{J}_1}) \cap G_s,$$

is also perfect of grade  $(\sum a_i) - n$ , where  $G_s = \cap H_k$  is defined as above. This can be established in exactly the same way as the perfection of  $G_s$  and  $P_\Delta + L_{\mathcal{J}_1}$  were. Let  $C_k = (P_\Delta + L_{\mathcal{J}_1}) \cap G_k$ , where  $G_0 = \langle 1 \rangle$ . We have already established that  $P_\Delta + L_{\mathcal{J}_1}$  is perfect of grade  $(\sum a_i) - n$ , so suppose that  $C_k$  is perfect. We have  $C_{k+1} = C_k \cap H_{k+1}$  and thus an exact sequence

$$0 \longrightarrow R/C_{k+1} \longrightarrow R/C_k \oplus R/H_{k+1} \longrightarrow R/(C_k + H_{k+1}) \longrightarrow 0.$$

Like the previous proofs,  $R/C_k$  and  $R/H_{k+1}$  we already know to be Cohen–Macaulay of depth  $(\sum a_i) - n$ , and  $R/(C_k + H_{k+1}) \cong (R/H_{k+1})/C_k$ , which is isomorphic to  $R/C_k$  for  $a_{k+1}$  decreased by 1, so it is Cohen–Macaulay of depth  $(\sum a_i) - n - 1$ . Therefore,  $R/C_{k+1}$  is Cohen–Macaulay of depth  $(\sum a_i) - n$ , so by induction,  $R/C_s = R/F_{1,\dots,1}$  is Cohen–Macaulay of depth  $(\sum a_i) - n$ .

Since  $F_{1,\dots,1} = P_\Delta + \langle x_{1,\dots,1,+,\dots,+} \rangle$  and  $x_{1,\dots,1,+,\dots,+}$  is a nonzerodivisor modulo  $P_\Delta$ , this implies that  $P_\Delta$  is perfect of grade  $1 - n + \sum a_i$ .  $\square$

**7.4. The Radicality of  $I_\Delta$ .** We now move from the prime ideal  $P_\Delta$  to the original ideal  $I_\Delta$ .

**Proposition 24.** *Let  $\Delta$  be a simplicial complex with two facets,  $\mathcal{J}_1, \mathcal{J}_2$  and let  $\mathcal{K}$  be a subset of  $\mathcal{J}_1 \cup \mathcal{J}_2$ . Then the ideal  $I_\Delta + L_{\mathcal{K}}$  is radical.*

*Proof.* We re-index so that  $\mathcal{J}_1 = \{1, \dots, s\}$  and  $\mathcal{J}_2 = \{r, \dots, n\}$ .

If  $\mathcal{K}$  contains  $\mathcal{J}_1$  or  $\mathcal{J}_2$ , this reduces to Lemma 20, so we suppose that  $\mathcal{K}$  contains neither  $\mathcal{J}_1$  nor  $\mathcal{J}_2$ . We will prove the result by principal radical systems. Define

$$\begin{aligned} F_{l_1,\dots,l_s} &= I_\Delta + L_{\mathcal{K}} + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid (i_1, \dots, i_s) \leq_{\text{revlex}} (l_1, \dots, l_s) \rangle \\ G_{l_1,\dots,l_s} &= I_\Delta + L_{\mathcal{K}} + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid i_j < l_j \text{ for some } j \leq s \rangle \end{aligned}$$

and let  $\mathcal{F} = \{F_{l_1,\dots,l_s} + G_{k_1,\dots,k_s}\}$ , the set of all sums of  $F$ 's and  $G$ 's. We claim that  $\mathcal{F}$  is a principal radical system.

If  $l = (l_1, \dots, l_s)$  is any sequence, let  $s(l)$  be the least  $l'$  such that  $l' >_{\text{revlex}} l$ . If  $j$  is the least  $j$  such that  $l_j \neq a_j$  then

$$s(l) = (1, \dots, 1, l_j + 1, l_{j+1}, \dots, l_s).$$

By definition,  $F_l + \langle x_{s(l)} \rangle = F_{s(l)}$ . Therefore,  $F_l + G_k + \langle x_{s(l)} \rangle = F_{s(l)} + G_k$ .

The following lemma will be the key to showing that  $\mathcal{F}$  is a principal radical system.

**Lemma 25.**  *$x_{l_1,\dots,l_s,+,\dots,+}$  is a nonzerodivisor modulo  $\text{rad } G_{l_1,\dots,l_s}$ .*

*Proof.* We will do this by computing the minimal primes over  $G_{l_1,\dots,l_s}$ , and showing that  $x_{l_1,\dots,l_s,+,\dots,+}$  is not in any of them. Let  $l' = (1, \dots, 1, l_r, \dots, l_s)$ . Then  $R/G_l \cong R/G_{l'}$  where the latter ring has the values of  $a_i$  decreased by  $l_i - 1$  for each  $i < r$ . Therefore, we can assume that  $l_i = 1$  for all  $i < r$ .

Suppose that  $l_i > 1$  for some  $i \geq r$ , without loss of generality, assume  $i = s$ . Then for each  $j_r, \dots, j_{s-1}$  and any  $j_s < l_s$

$$x_{+, \dots, +, j_r, \dots, j_s, +, \dots, +} \in G_l.$$

Since  $I(A_{\mathcal{J}_2}) \subset G_l$ , any prime containing  $G_l$  must either contain  $L_{\{r, \dots, s\}}$  or  $x_{+, \dots, +, j_r, \dots, j_s, j_{s+1}, \dots, j_n}$  for all  $j_s < l_s$ .

Let  $H_{l_1, \dots, l_s} = G_{l_1, \dots, l_s} + \langle x_{+, \dots, +, i_r, \dots, i_n} \mid i_j < l_j \text{ for some } r \leq j \leq s \rangle$ . The previous paragraph implies that any prime containing  $G_{l_1, \dots, l_s}$  either contains  $L_{\{r, \dots, s\}}$  or contains  $H_{l_1, \dots, l_s}$ . Since  $R/H_l$  is isomorphic to  $R/(I_\Delta + L_{\mathcal{K}})$  with  $a_i$  decreased by  $l_i - 1$  for each  $i$ . Therefore, to show that  $x_{l_1, \dots, l_s, +, \dots, +}$  is not in a minimal prime over  $H_{l_1, \dots, l_s}$  is the same as showing that  $x_{1, \dots, 1, +, \dots, +}$  is not in a minimal prime over  $I_\Delta + L_{\mathcal{K}}$ .

The minimal primes over  $I_\Delta + L_{\mathcal{K}}$  are either  $P_\Delta + L_{\Delta, i}$  for some  $i \in (\mathcal{J}_1 \cap \mathcal{J}_2) \setminus \mathcal{K}$  or  $I_\Delta + L_{\mathcal{J}_1 \setminus i_1} + L_{\mathcal{J}_2 \setminus i_2}$  where  $i_1, i_2 \notin \mathcal{K}$ . It is clear that  $x_{1, \dots, 1, +, \dots, +}$  is not in any of these ideals.

On the other hand, we must show that  $x_{1, \dots, 1, +, \dots, +}$  is not in any of the minimal primes over  $G_l + L_{\{r, \dots, s\}}$ . Because this ideal contains  $L_{\{r, \dots, s\}}$ , it can be expressed, in  $S_\Delta$  as  $I_1 + I_2$  where  $I_1 \subset \mathbb{K}[X_{i_1, \dots, i_s, +, \dots, +}]$  and  $I_2 \subset \mathbb{K}[X_{+, \dots, +, i_r, \dots, i_n}]$ . Therefore, we need only consider the minimal primes over

$$I(A_{\mathcal{J}_1}) + L_{\mathcal{K} \cap \mathcal{J}_1} + L_{\{r, \dots, s\}} + \langle x_{i_1, \dots, i_s, +, \dots, +} \mid i_j < l_j \text{ for some } j \leq s \rangle.$$

The effect of the last summand is only to reduce each  $a_i$  by  $l_i - 1$ , so we may assume that this term is 0. Then we are left with  $I(A_{\mathcal{J}_1}) + L_{\mathcal{K} \cap \mathcal{J}_1} + L_{\{r, \dots, s\}}$ , whose minimal primes are contained in  $I(A_{\mathcal{J}_1}) + L_{\mathcal{J}_1 \setminus i} + L_{\mathcal{J}_1 \setminus j}$  where  $i \notin \mathcal{K}$  and  $j < r$ . Thus  $x_{1, \dots, 1, +, \dots, +}$  is not in any minimal prime over  $G_l + L_{\{r, \dots, s\}}$ .

This completes the proof of the lemma, so  $x_{l, +, \dots, +}$  is a nonzerodivisor modulo  $G_l$ .  $\square$

Since  $G_{s(l)} \supsetneq F_l$  for any  $l$ ,  $F_l + G_k = G_k$  whenever  $k >_{\text{revlex}} l$ . Moreover,

$$G_l + F_l = G_l + \langle x_{l_1, \dots, l_s, +, \dots, +} \rangle$$

so by our lemma, if  $k >_{\text{revlex}} l$   $F_l + G_k$  satisfies condition 1 of theorem 19.

On the other hand, if  $k \leq l < (a_1, \dots, a_n)$ , recall that

$$G_k + F_{s(l)} = G_k + F_l + \langle x_{s(l), +, \dots, +} \rangle.$$

$G_{s(l)} \supsetneq G_k + F_l$ , and  $x_{s(l), +, \dots, +} G_{s(l)} \subset G_k + F_l$ . Thus, since  $x_{s(l), +, \dots, +}$  is a nonzerodivisor modulo  $\text{rad } G_{s(l)}$  by the lemma,  $G_k + F_l$  satisfies condition 2 of theorem 19.

Finally,

$$F_{a_1, \dots, a_n} = I_\Delta + L_{\mathcal{K}} + L_{\mathcal{J}_1} = I(A_{\mathcal{J}_2}) + L_{\mathcal{K}} + L_{\mathcal{J}_1}$$

which is radical by Lemma 20.

Therefore,  $\mathcal{F}$  is a principal radical system and  $I_\Delta + L_{\mathcal{K}}$  is radical.  $\square$

**Theorem 26.** *If  $\Delta$  has three or fewer facets then  $I_\Delta$  is a radical ideal.*

*Proof.* If  $\Delta$  has one or two facets, this has been proven in Lemma 20 and Proposition 24, so we may assume that  $\Delta$  has three facets.

This proof is very similar to that of Proposition 24. Re-index so that  $\mathcal{J}_1 = \{1, \dots, s\}$ , and let

$$\begin{aligned} F_{l_1, \dots, l_s} &= I_\Delta + \langle x_{i_1, \dots, i_s, +, \dots, +} \mid (i_1, \dots, i_s) \leq_{\text{revlex}} (l_1, \dots, l_s) \rangle \\ G_{l_1, \dots, l_s} &= I_\Delta + L_K + \langle x_{i_1, \dots, i_s, +, \dots, +} \mid i_j < l_j \text{ for some } j \leq s \rangle. \end{aligned}$$

Define  $\mathcal{F} = \{F_{l_1, \dots, l_s} + G_{k_1, \dots, k_s}\}$ , the set of all sums of  $F$ 's and  $G$ 's. We claim that  $\mathcal{F}$  is a principal radical system.

We will prove below that  $x_{l, +, \dots, +}$  is a nonzerodivisor modulo  $\text{rad } G_l$  and now we show how that will imply the theorem.

As in the previous proof,  $F_l + G_k + \langle x_{s(l)} \rangle = F_{s(l)} + G_k$ , so so for if  $k >_{\text{revlex}} l$   $F_l + G_k$  satisfies condition 1 of theorem 19. Moreover, if  $k \leq l < (a_1, \dots, a_n)$ ,

$$G_k + F_{s(l)} = G_k + F_l + \langle x_{s(l), +, \dots, +} \rangle.$$

$G_{s(l)} \supsetneq G_k + F_l$ , and  $x_{s(l), +, \dots, +} G_{s(l)} \subset G_k + F_l$ . Thus, since  $x_{s(l), +, \dots, +}$  is a nonzerodivisor modulo  $\text{rad } G_{s(l)}$  as we will show below,  $G_k + F_l$  satisfies condition 2 of theorem 19.

Finally,  $F_{a_1, \dots, a_s}$  is radical because it is  $I_{\mathcal{J}_2, \mathcal{J}_3} + L_{\mathcal{J}_1}$  which is radical by Proposition 24.

Therefore, the theorem will be completed with the proof of the following lemma

**Lemma 27.**  $x_{l_1, \dots, l_s, +, \dots, +}$  is a nonzerodivisor modulo  $\text{rad } G_{l_1, \dots, l_s}$ .

*Proof.* Again, we prove this by computing the minimal primes over  $G_{l_1, \dots, l_s}$  and showing that  $x_{l_1, \dots, l_s, +, \dots, +}$  is not in any of them.

Suppose  $Q$  is a minimal prime over  $G_{l_1, \dots, l_s}$  which contains  $L_{\mathcal{J}_1 \cap \mathcal{J}_2} + L_{\mathcal{J}_1 \cap \mathcal{J}_3}$ . Since  $G_{l_1, \dots, l_s} + L_{\mathcal{J}_1 \cap \mathcal{J}_2} + L_{\mathcal{J}_1 \cap \mathcal{J}_3}$  can be expressed as  $I_1 + I_2$  where  $I_1 \subset \mathbb{K}[X_{i_1, \dots, i_s, +, \dots, +}] = S_{\mathcal{J}_1}$  and the generators  $I_2$  have none of those variables in them, we can show that  $x_{l_1, \dots, l_s, +, \dots, +} \notin Q$  by showing that it is not in any minimal prime over  $I(A_{\mathcal{J}_1}) + L_{\mathcal{J}_1 \cap \mathcal{J}_2} + L_{\mathcal{J}_1 \cap \mathcal{J}_3}$ , which is clear.

The second case is when  $Q$  is a minimal prime over  $G_{l_1, \dots, l_s}$  which contains  $L_{\mathcal{J}_1 \cap \mathcal{J}_3}$  but not  $L_{\mathcal{J}_1 \cap \mathcal{J}_2}$ . Then it must also contain  $L_{\mathcal{J}_2 \cap \mathcal{J}_3}$  and  $P_{\{\mathcal{J}_1, \mathcal{J}_2\}}$  by Corollary 14. As in the previous paragraph,

$$G_{l_1, \dots, l_s} + P_{\mathcal{J}_1, \mathcal{J}_2} + L_{\mathcal{J}_1 \cap \mathcal{J}_3} + L_{\mathcal{J}_2 \cap \mathcal{J}_3}$$

can be expressed as  $I_1 + I_2$  where  $I_1 \subset S_{\{\mathcal{J}_1, \mathcal{J}_2\}}$  and the generators  $I_2$  have none of those variables in them. Thus we can show that  $x_{l_1, \dots, l_s, +, \dots, +} \notin Q$  by showing that it is not in any minimal prime over  $P_{\{\mathcal{J}_1, \mathcal{J}_2\}} + G_{l_1, \dots, l_s} + L_{\mathcal{J}_3}$ . Since this case was covered in Lemma 25, we refer to that proof.

The final case is that in which  $Q$  is a minimal prime over  $G_{l_1, \dots, l_s}$  and contains neither  $L_{\mathcal{J}_1 \cap \mathcal{J}_2}$  nor  $L_{\mathcal{J}_1 \cap \mathcal{J}_3}$ . Thus, it cannot contain  $L_{\mathcal{J}_1 \cap \mathcal{J}_3}$  either and must contain  $P_\Delta$ . Moreover, if  $i \in \mathcal{J}_1 \cap \mathcal{J}_2$  and  $l_i > 1$ , then we can re-index so  $i = s$  and  $\mathcal{J}_2 = \{r, \dots, n\}$ . As in the proof of Lemma 25,  $G_{l_1, \dots, l_s}$  contains  $x_{+, \dots, +, j_r, \dots, j_s, +, \dots, +}$  for all  $j_r, \dots, j_{s-1}$  and any  $j_s < l_s$ . Since  $Q$



does not contain  $L_{\mathcal{J}_1 \cap \mathcal{J}_2}$  it must be the case that  $x_{+, \dots, j_r, \dots, j_n} \in Q$  as long as  $j_s < l_s$ . Therefore,  $Q$  must contain

$$H_{l_1, \dots, l_s} = P_\Delta + \langle x_{i_1, \dots, i_n} \mid i_j < l_j \text{ for some } j \leq s \rangle.$$

(Notice that this ideal was defined as  $G_{l_1, \dots, l_s}$  in the proof of Theorem 22.) Since  $R/H_{l_1, \dots, l_s}$  is isomorphic to  $R/P_\Delta$  where each  $a_i$  has been reduced by  $l_i - 1$ . Therefore,  $H_{l_1, \dots, l_s}$  is prime and  $x_{l_1, \dots, l_s, +, \dots, +}$  is not in it.

We have shown that  $x_{l_1, \dots, l_s, +, \dots, +}$  is not in any minimal prime over  $G_{l_1, \dots, l_s}$  and hence is a nonzerodivisor modulo its radical.  $\square$

This completes the proof that  $I_\Delta$  is radical if  $\Delta$  has fewer than three facets.  $\square$

## 8. CONJECTURES, EXAMPLES, AND NOTES ON COMPUTATION

**8.1. An Example In Which  $I_\Delta$  Is Not Radical.** It is not true that for any  $\Delta$ ,  $I_\Delta(A)$  is radical. Any time  $Q_\Delta \neq J_\Delta$ , we know that

$$x_{+, \dots, +} \cdot Q_\Delta \subset \text{rad}(I_\Delta);$$

however, this will not always be contained in  $I_\Delta$ . For example, when

$$\Delta = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$$

it can be shown computationally that

$$x_{+, +, +, +}(x_{1, 1, +, +}x_{+, +, 1, 1} - x_{1, +, 1, +}x_{+, 1, +, 1}) \notin I_\Delta.$$

In this case, it turns out that the primary decomposition is still accessible, and we give a computation of it in the case  $a_i = 2$ . Let

$$\begin{aligned} Q_1 &= P_{\{1, 2\}, \{2, 4\}} + P_{\{1, 3\}, \{3, 4\}} + \langle x_{i, +, +, +} \rangle + \langle x_{+, +, +, l} \rangle \\ Q_2 &= P_{\{1, 2\}, \{1, 3\}} + P_{\{2, 4\}, \{3, 4\}} + \langle x_{+, j, +, +} \rangle + \langle x_{+, +, k, +} \rangle \\ Q_3 &= I_\Delta + \langle x_{i, +, +, +}^2, x_{+, j, +, +}^2, x_{+, +, k, +}^2, x_{+, +, +, l}^2, x_{+, +, +, +}^2 \rangle \end{aligned}$$

It can be verified using Macaulay 2 [GS] that

$$I_\Delta = P_\Delta \cap Q_1 \cap Q_2 \cap Q_3.$$

**8.2. Two Conjectures.** Section 7 has exclusively dealt with the case in which  $\Delta$  has three or fewer facets. We offer the following conjectures which have been borne out in all the examples which our computers have been able to accomplish.

**Conjecture 28.** *If  $\Delta$  is any simplicial complex,*

$$K_\Delta + Q_\Delta = P_\Delta,$$

*which is a prime and perfect ideal of grade  $1 - n + \sum a_i$ .*

We have proven this result in the case in which  $\Delta$  has three or fewer facets. Moreover, we have shown that  $\text{rad}(K_\Delta + Q_\Delta)$  is prime in Theorem 7, which should be seen as good evidence for the primality of the ideal.

The second conjecture deals with the radicality of  $I_\Delta$ .

**Conjecture 29.** *Let  $\Delta$  be any simplicial complex.  $I_\Delta$  is a radical ideal if and only if  $Q_\Delta = J_\Delta$ .*

**8.3. Notes on Computation.** Finally, we discuss the computational aspects of experimenting with these families of ideals. All computations should be done in  $T_\Delta$  because it reduces the number of variables in the polynomial ring. This reduction is especially noticeable when some of the  $a_i > 2$ . A side benefit is that the relations are usually easier to decipher when they are expressed in the variables of  $S_\Delta$ . In fact, these were the reasons that first attracted me to change variables.

If  $a_i = 2$  for all  $i$ , then we are in a position to decompose  $I_\Delta$  when  $\Delta$  has fewer than four vertices ( $n \leq 4$ ), and can do some cases with five or six vertices. After that point, the only  $\Delta$ 's for which  $I_\Delta$  can be decomposed have two facets.

When  $a_i = 2$  it is also possible to compute a free resolution for  $P_\Delta$  for some cases until  $n = 5$ . After that, the problem again becomes insurmountable.

If we allow  $a_i > 2$ , both problems become very difficult very fast. The decomposition can be checked by using Theorem 15, and intersecting the minimal primes. Computing a free resolution also becomes computationally impossible very fast. For the simplest  $\Delta$  with three facets,  $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ , a free resolution cannot be computed when  $a_i = 3$  for each  $i$ .

## REFERENCES

- [BV88] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988. [MR 89i:13001](#)
- [GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*.
- [Hà02] Huy Tài Hà, *Box-shaped matrices and the defining ideal of certain blowup surfaces*, J. Pure Appl. Algebra **167** (2002), no. 2-3, 203–224. [MR 2002h:13020](#)
- [Mat99] F. Matúš, *Conditional independences among four random variables. III. Final conclusion*, Combin. Probab. Comput. **8** (1999), no. 3, 269–276. [MR 2000i:68176](#)
- [Stu02] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2002. [MR 2003i:13037](#)

UNIVERSITY OF CALIFORNIA, BERKELEY  
*E-mail address:* `kirkup@math.berkeley.edu`